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Initial state parton showers beyond leading order

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Abstract: We derive a new method for initial-state collinear showering in Monte-Carlo event generators which is based on the use of unintegrated parton correlation functions. Combined with a previously derived method for final-state showering, the method solves the problem of treating both the hard scattering and the evolution kernels to be used in arbitrarily non-leading order. Although we only treat *collinear* showering, so that further extensions are needed for QCD, we have discovered several new results: (1) It is better to generate exact parton kinematics in the hard scattering rather than with the subsequent parton showering, and similarly at each step of the showering. (2) Parton showering is then done conditionally on the exact energy-momentum of the initiating parton. (3) We obtain a factorization for structure functions in terms of parton correlation functions so that parton kinematics can be treated exactly from the beginning. (4) We obtain two factorization properties for parton correlation functions, one in terms of ordinary parton densities and one, suitable for event generation, in terms of parton correlation functions themselves.

Keywords: QCD, NLO Computations, Deep Inelastic Scattering, Jets.

Contents

1. Introduction

Monte-Carlo event generators (MCEGs) provide a very powerful approach to implementing perturbative QCD to predict many observables in hard processes. However, despite a considerable amount of work — see the paper by Frixione and Webber [1] and the references reviewed there — current algorithms do not systematically extend beyond an improved leading-logarithm approximation. This is in contrast to factorization theorems for hard inclusive processes, where it is known how both hard-scattering coefficients and evolution kernels may be computed, in principle, to arbitrarily high order in perturbation theory. As a computational tool, event generators not only provide estimates of the exclusive components of cross sections, but they have the important property that event generation needs a time roughly linear in the number of particles. In contrast, the computation of Feynman graphs for cross sections involving N particles needs resources proportional to roughly $N!$.

In this paper, we provide a solution to the problem of non-leading corrections in MCEGs when initial-state collinear showering is involved. Combined with a method one of us developed [2] for final-state collinear showering, this allows a complete treatment for deep-inelastic scattering, for example. Our treatment in this paper is restricted to collinear showering, so it applies as it stands only to non-gauge theories. However, even though substantial extension of the method will be needed to handle QCD, with its soft gluon emission, our work solves a number of important conceptual problems whose solution is also needed in QCD.

We will give a more detailed account of the problems we are concerned about in section 2. Here we will simply remark that the algorithms in standard MCEGs are based on the factorization theorem [3] for inclusive hard processes, whereas MCEGs concern themselves with the exclusive components of cross sections. Thus the standard factorization theorem does not really provide a fully satisfactory foundation for MCEGs; a more general and powerful theorem is needed. This can be seen in several ways:

- The proofs [3] of standard factorization explicitly involve sums over unobserved final states, but MCEGs resolve the full final state.
- Approximations on the kinematics of internal partons are made that become invalid if the exact final state is observed: the approximations violate exact momentum conservation. Moreover, as we will show in section 2, the approximations interfere with the factorized structure of probabilities (or weights) that enables the stochastic generation of events to take a time linear in event size.
- In standard factorization, cross sections that are infra-red and collinear safe are approximated by singular distributions like

$$(1 + \alpha_s A)\,\delta(x) + \alpha_s B \left(\frac{1}{x}\right)_+ + \cdots . \tag{1.1}$$

  While such approximations can be valid *when integrated with a smooth function*, they are not valid when a fully differential cross section is under consideration.

- Although the previous problem can be alleviated by some kind of resummation, it is preferable to have an improved factorization theorem from the beginning.

The most fundamental part of the proof of factorization, the analysis of the momentum-space regions that give leading contributions to the cross section, is indeed generally applicable, and so it does give a correct starting point for deriving algorithms for MCEGs. What is at issue are the further steps, and in particular the approximations, that are used to obtain the standard form of factorization; these approximations are only appropriate when there is an inclusive sum over the hadronic final state.

Our approach solves the problems by working with parton correlation functions that are defined without an integral over parton kinematics, unlike conventional parton densities. This avoids the need to use approximated parton kinematics in places where the approximation is invalid. Moreover, we will also work with parton correlation functions and cross sections that are defined without an integral over hadronic final states (or, more conveniently, with an arbitrary weight function of the hadronic final state). This gives us the appropriate tools for systematically and precisely treating the actual physical situations which MCEGs aim to approximate.

Of course, parton correlation functions are more complicated objects than parton densities, so without further information a lot of predictive power is lost. We solve this issue with extra factorization theorems for the parton correlation functions at large parton virtuality; the showering algorithm is what implements these.

At low parton virtuality, the information in parton correlation functions corresponds to the information in non-perturbative hadronization models that are present in conventional MCEGs. There is no gain or loss of predictive power here.

We will find that we can explicitly represent all probabilities and cross sections for a MCEG in terms of integrals over parton correlation functions, and hence in terms of explicit operator matrix elements. The systematic extension to non-leading order and non-leading logarithms then becomes straightforward, with the use of the subtractive methods advocated by us in [2, 4].

Related work is in papers by Watt, Martin and Ryskin [5, 6] who have proposed the use of what they call "doubly unintegrated parton distributions" (DUPDFs). This concept coincides with our "parton correlation functions". Their primary motivation is the same as ours: the need to treat parton kinematics exactly in certain observables in a way that is fully compatible with ordinary parton densities. Their work treats QCD, so it is of direct phenomenological significance. However, it is also restricted to a kind of leading approximation; they obtain a result for the DUPDFs from an examination of the last step of the evolution. This corresponds to one of our extra factorization theorems for parton correlation functions, in its leading-order approximation when the parton kinematics are strongly ordered. Our aims are more ambitious even if we have not achieved them in QCD so far: we have obtained a formalism that can be applied to arbitrarily non-leading corrections and in the context of a MCEG. To this end, we have provided exact operator definitions of our distributions. In QCD, as we will explain, it is necessary to deal with the issues symptomized by the gauge invariance problem of the definition of the pCfs: Where

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should the Wilson lines be? In a leading logarithm approximation, these issues are readily evaded by inserting appropriate cutoffs, without examining exactly how these are to be implemented in an exact operator definition.

The organization of the paper is the following. After explaining the conceptual issues that concern us in section 2, we describe our model theory in section 3. Then in section 4 we derive our new factorization for DIS structure functions in terms of parton correlation functions. In section 5, we obtain factorization theorems for parton correlation functions; these give both the showering algorithm and the computation of parton correlation functions in terms of ordinary parton densities (as in the $\overline{\text{MS}}$ scheme). In section 6 we show an algorithm for a MCEG that implements our factorization formulas. Then in section 7 we give explicit calculations of the next-to-leading order (NLO) hard scattering for deep-inelastic scattering. Finally, in section 8, we give our conclusions and an indication of the approach we propose to use to extend our results to full QCD.

2. Rationale for a new algorithm

The overall structure of a MCEG is that each event is generated by the following steps:

1. Generate a hard scattering. This results in a set of partons and values of their momenta (or of some scalar variables that can be used to determine the momenta).

2. For each available parton, either choose not to shower it, or generate a branching, i.e., a set of new partons.

3. Go back to the previous step, applying it to any new partons that have been generated.

This general structure corresponds to a factorization theorem, and the generated events can be given a Feynman-graph-like structure, as in figure 1 for deep-inelastic scattering (DIS). The generation of the hard scattering is represented by the vertex labeled q, and the resulting partons are represented by the attached lines. The branchings of these partons are represented by the vertices at the other ends of the lines. The arrows label momentum flow with respect to a corresponding Feynman graph vertex. The lines with no vertex at the outgoing end are final-state particles, and the one line, labeled P, with no vertex at the incoming end is the target particle. Some generalization is needed in the case of angular ordered coherent branching, but this does not affect the overall ideas. Notice that the lines represent both the generated partons and the flow of information between different units of computation, at the vertices.

The key property that gives a straightforward algorithm is that at each step the showering probability for generating partons depends on a fixed number of variables that correspond to the single parent parton of that step. The showering probability is, of course, a conditional probability.[1] Since at each step we have several partons, each of which needs to

[1]In a subtractive formalism beyond leading order, such as we will use, correction terms to the cross section may sometimes be negative, so we will allow the generation of weighted events, some with negative

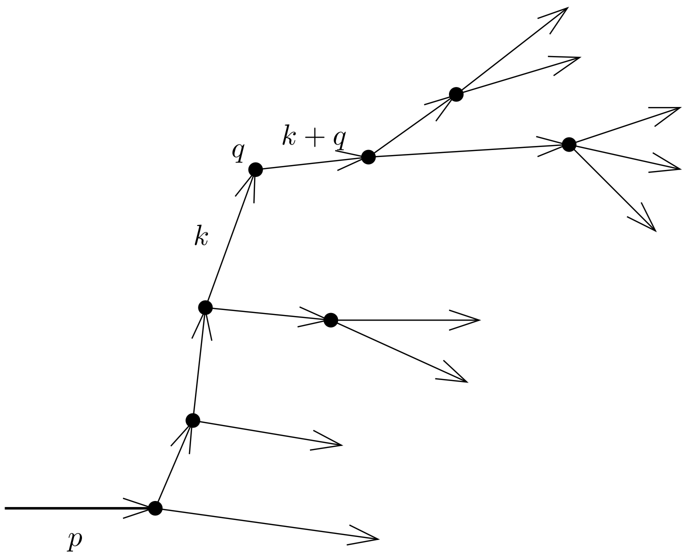

Figure 1: Generation and structure of event from MCEG for deep-inelastic scattering. One vertex has four attached lines; it would arise from a possible NLO branching.

be showered, the showering probabilities for the different partons should be independent; this corresponds to a factorization theorem. What is needed to allow a clear systematic discussion of non-leading corrections is a definite operator matrix element expression for the probability for showering; any non-trivial communication between the showers, as exhibited by a non-factorization, will tend to preclude the possibility of a matrix element formulation. Without a systematic factorization,[2] so that different showers are independent, it is hard to give a systematic treatment for the production of arbitrarily many particles.

As we will now show, the most obvious formulation for the algorithm, founded on the parton model approximation to factorization, actually violates some of these properties because of its treatment of parton kinematics. This is what motivates us to find a more general approach.

For ease of presentation, we shall work with a model theory ϕ^3 in space-time dimension $n = 6$ throughout the paper, the methods immediately generalize to any nongauge theory. The details of the model will be described in section 3.

We consider DIS, whose kinematics are shown in figure 2. There q is the photon momentum and P the hadron momentum. We work in the virtual-photon-hadron center-of-mass frame, using light-cone coordinates, $p^{\pm} = (p^0 \pm p^z)/\sqrt{2}$, where, up to corrections

weights. For this purpose, we must generalize the use of the term "probability". Instead of a conditional probability, we will need what we might term a conditional weight, normalized like a probability so that its integral is unity.

[2]Of course, factorization does not hold as an exact statement. It is only true as an approximation for certain important classes of kinematic region. Also, when higher-order corrections are included, we obtain not an actually factorizing formula for the cross section but a sum of factorized terms, both for the hard scattering and for the parton branching.

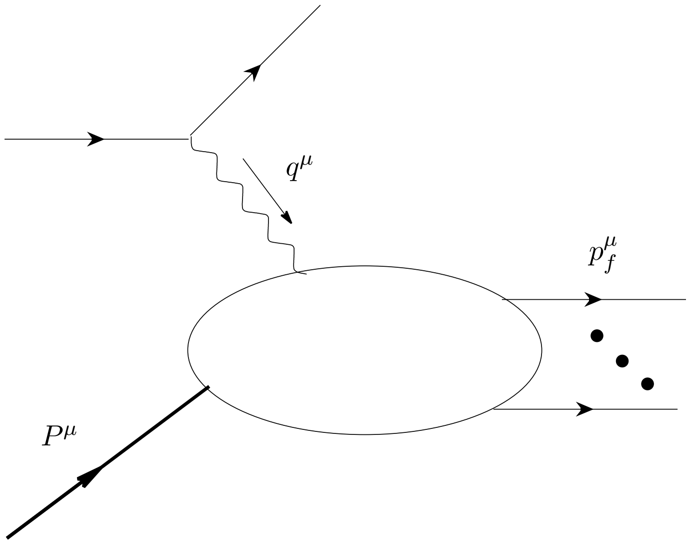

Figure 2: Deep inelastic scattering.

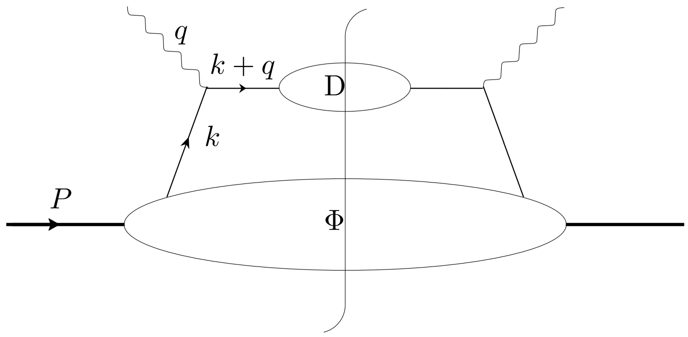

Figure 3: "Handbag" diagram for DIS.

power-suppressed at large Q,

$$q^\mu = \left(-xP^+, \frac{Q^2}{2xP^+}, \vec{0}_T\right), \tag{2.1}$$

$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \vec{0}_T\right), \tag{2.2}$$

$$x \equiv \frac{Q^2}{2P \cdot q} \qquad \text{and} \qquad Q^2 \equiv -q^2 \,. \tag{2.3}$$

The parton-model approximation is obtained from handbag diagrams, as in figure 3. Notice that we have included a bubble representing the full showering and hadronization of the struck quark. Up to a normalization factor, which will not concern us here, the value of the handbag is

$$F_2 = \frac{Q^2}{2\pi} \int \frac{d^6k}{(2\pi)^6}\, \Phi(k,P)\, D(k), \tag{2.4}$$

where Φ and D correspond to sums over the correspondingly labeled subgraphs in figure 3, with an integral over all final states. The parton-model approximation [7] applies when

k has low transverse momentum and virtuality compared with Q and when the struck quark also has virtuality much less than Q. Then in the hard scattering, which in this case is a trivial factor of unity, we can replace the incoming and outgoing quarks by on-shell momenta

$$\begin{aligned} k &\simeq (xP^+, 0, \vec{0}_T)\,, \\ k+q &\simeq (0, q^-, \vec{0}_T) = \left(0, \frac{Q^2}{(2xP^+)}, \vec{0}_T\right). \end{aligned} \tag{2.5}$$

However, this approximation cannot be made in the two nonperturbative blobs Φ and D, but rather the small components of momentum must be integrated over. This leads to

$$F_2 \simeq \left[xP^+ \int \frac{dk^-\, d^4\vec{k}_T}{(2\pi)^6} \Phi(xP^+, k^-, \vec{k}_T, P) \right] \left[\frac{2q^-}{2\pi} \int dl^+ D(l^+, q^-, \vec{0}_T) \right]. \tag{2.6}$$

Up to a normalization, the first factor in square brackets corresponds to the usual definition of a parton density, $f(x)$, as the matrix element of a light-front operator. The second factor is the integral of the fragmentation function D over final states, again up to a normalization; this integral is over the discontinuity of a propagator, so that the factor is equal to unity if the integral is convergent.[3]

Notice that in the Φ and D factors in the parton-model approximation we have replaced the large components k^+ and $k^- + q^-$ of the external momenta by the same approximate values xP^+ and q^- that are given in eq. (2.5); it is only the small components that we leave unchanged.[4] We have therefore replaced the final states of Φ and D by final states with different momenta. For the inclusive structure function F_2, this is satisfactory, since we are only concerned with the numerical values of the factors in eq. (2.6), not with the detailed structure of the final state. For this case, errors in the approximation at large parton virtuality are properly compensated by the usual correctly derived higher-order corrections to the hard scattering.

But the situation is different in a MCEG, as we will now see. In the first step of a MCEG, a hard-scattering configuration is generated with a corresponding value of parton momentum k. Since the showering has not yet been performed, it is natural and conventional to assign to this momentum the naive parton model value, as in eq. (2.5). The cross section is expressed in terms of the parton density $f(x)$, which, as in the first factor of eq. (2.6), is defined in terms of an integral over parton momentum k with the plus component fixed: $k^+ = xP^+$. However the true value is different:

$$k^+ = xP^+ + \frac{M^2 + k_T^2}{2(k^- + q^-)}\,, \tag{2.7}$$

where M is the invariant mass of the outgoing parton.

[3]There are, in fact, ultra-violet convergence problems in these approximated integrals, associated with the need for NLO corrections to the parton model; the divergences can be dealt with either by a cutoff or by renormalization [3]. But this issue will not affect this part of our discussion.

[4]We have also replaced the parton transverse momentum in the fragmentation by zero. But this is simply equivalent to a small Lorentz transformation.

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Thus there is a mismatch between the true value and the initially generated value of k^+ that is used to define the parton density. This can be a large mismatch, since the virtualities and transverse momentum range up to order Q, at least. The parton-model approximation replaces the final state in blob Φ by a final state with a different momentum.

Since in a MCEG we resolve the structure of the final state, the value of k has to be reassigned after showering to correspond to the true final state. The use of the word "reassigned" in the previous sentence unmasks a conceptual shift that is critical in correctly analyzing the algorithm in a MCEG. There are in fact (at least) three very different objects to which we can give the name k. Two correspond to the dummy variables of integration in one or other of the two formulas (2.4) and (2.6) for the handbag diagram; within a Monte-Carlo implementation each of these variables can be treated as a random variable which has a single definite (Lorentz-vector) value in each event. The third object called k is a storage location in the computer program; its value can change during the generation of a single event. Such a reassignment of the computer variable changes the correspondence between the computer variable and the random variable(s) used to define probabilities.

Branching probabilities in initial-state showering are conditional probabilities given the value of k, and in particular given the value of k^+. But the value of k^+ used in the initial-state showering changes during showering, in a way that depends on the shower history of the event. The same applies for the value of $k^- + q^-$ in the final-state shower. This makes the fundamental basis of a conventional MCEG and of its independent showering probabilities hard to quantify accurately enough to allow a transparent derivation of higher-order corrections.

A useful characterization of the difficulty associates it with the momentum-conservation delta function at the photon vertex. Let l be the momentum of the outgoing quark. Then the standard parton-model approximation replaces

$$\delta^{(6)}(k + q - l) \tag{2.8}$$

by

$$\delta(k^+ + q^+)\, \delta(q^- - l^-)\, \delta^{(4)}(\vec{l}_T)\,. \tag{2.9}$$

These are good approximations to each other only if they are integrated with a sufficiently slowly varying function, as is the case for the inclusive cross section. But when we resolve the full final state, the two delta functions are not good approximations to each other; they are infinitely different.

Therefore we propose that the fundamental factorization formula for a MCEG should involve unintegrated parton correlation functions, and that showering then involve probabilities conditional on k and $k + q$ which have already been assigned their exact values. While the approximated momenta will still make their appearance, it will be in a different and controllable fashion.

By using a factorization formula which does not contain ordinary parton densities we have also removed the DGLAP evolution kernels, which appear to form an essential part of the showering algorithm. Other quantities will take over their role.

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We will find it convenient not to generate the exact parton momenta at one step, but to reformulate in several substeps the first step of the algorithm that we gave at the beginning of this section. Thus we can generate the hard scattering as follows:

1.a Generate the hard scattering with conventional massless external partons, and conventional parton densities.[5]

1.b Generate, independently, the virtualities and transverse momenta of the partons according to some suitable approximate distributions.

1.c Reassign the parton momenta to their exact values according to a defined prescription.

1.d Reweight (or appropriately veto) the event so that the probability/weight corresponds to the correct distribution in terms of parton correlation functions. The values of the parton correlation functions will have been tabulated as the result of a separate calculation.

A similar procedure is applied at each step of parton showering. The motivation of this procedure is that the factorization formula in terms of unintegrated parton correlation functions involves more variables and more complicated objects than standard factorization. So it is simpler to base the event generation on ordinary parton densities and then correct the result to correspond to the new factorization formula.

Although the reassignment of parton momenta and the reweighting are superficially similar to those in other proposals, the important point is that we propose a change in the logic. Previously the hard scattering and the branchings steps in showering generated approximate parton momenta. The reassignment of parton momenta to their exact values occurred *after* showering of these partons. If we then try to use a reweighting method to treat higher-order corrections for both the hard scattering and for showering probabilities, there is overlap between the parts of the event to which different reweightings are applied.

We instead do the reassignment of parton momenta *before* the showering. All the probability distributions for the actual parton momenta are distributions as functions of the exact momenta. Therefore the branching probabilities for different partons can be taken as strictly independent of each other. Moreover the reweighting is not to make the differential cross section correspond to the standard NLO calculation of the cross section but rather to make it correspond to our new factorization formula in terms of parton correlation functions. Of course, the new formula will include the information in the previous calculations, but it can potentially include even higher order corrections. Most importantly, the application of reweighting to the hard scattering or one branching does not overlap logically with later branchings. Thus exactly similar methods can be applied to both the hard scattering and to parton branching, so that both can be systematically treated to arbitrary order.

[5]The parton densities should be the most accurate known parton densities, rather than just the approximation with LO evolution.

Our procedure is a generalization of the work by one of us in [2] for final-state showering. The new feature is in a certain sense concerned with the directions of cause and effect. In e^+e^--annihilation we generate a hard scattering given the incoming leptons. We shower the partons and readjust their kinematics by some chosen prescription to satisfy momentum conservation.

But with an initial-state hadron, this procedure requires us to readjust the kinematics of the *incoming* parton whose value of k^+ was used to determine the cross section from the parton density. The transformation of the kinematics to get correct parton momenta inevitably changes the value of k^+ — see eq. (2.7). This invalidates the value of the non-perturbative parton density used at the initial step of computing the hard-scattering probability. The corresponding non-perturbative probability in e^+e^--annihilation is the unit probability for a parton to shower.

3. The model for DIS

Our method in its present form works for any non-gauge theory. But for simplicity of presentation, we work with ϕ^3 theory in 6 space-time dimensions. The lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{6}g\left(\frac{\mu^2 e^\gamma}{4\pi}\right)^{\epsilon/2}\phi^3 + \mathcal{L}_{ct}\,, \\ \mathcal{L}_{ct} &= \frac{1}{2}\delta Z_2(\partial\phi)^2 - \frac{\delta m^2\phi^2}{2} - \frac{1}{6}\delta g\phi^3 - \delta h\phi\,. \end{aligned} \tag{3.1}$$

For regularization of ultra-violet divergences we use a space-time dimension $n = 6 - 2\epsilon$. We associate a factor $\bar{\mu}^\epsilon \equiv [\mu^2 e^\gamma/(4\pi)]^{\epsilon/2}$ with the coupling so that $\overline{\text{MS}}$ renormalization is implemented by pure pole counterterms. We use $\overline{\text{MS}}$ renormalization for all except the renormalization of tadpole graphs, and for those we define the $\delta h\phi$ counterterm [8] by requiring the sum of the tadpoles and their counterterms to vanish, i.e., by the renormalization condition $\langle 0|\phi|0\rangle = 0$. We define a bare parton field as $\phi_0 = Z^{1/2}\phi$.

Our model for DIS consists of the exchange of a weakly interacting boson, A, for which we add an extra term to the interaction $\mathcal{L}_s = e_0 A\phi_0^2/2$ and a corresponding counterterm. As in [2], we unify the calculation of all cross sections in a weighted cross section

$$\begin{aligned} \sigma[W] &= K\sum_f \langle P|j(0)|f\rangle\langle f|j(0)|P\rangle(2\pi)^6\delta^{(6)}(q+P-p_f)\,W(f) \\ &= K\sum_f W(f)\int d^6y\, e^{iq\cdot y}\langle P|j(y)|f\rangle\langle f|j(0)|P\rangle\,, \end{aligned} \tag{3.2}$$

where the weighting function $W(f)$ is an arbitrary smooth function of the final state. The current $j = e\frac{1}{2}[\phi^2] = e_0\frac{1}{2}\phi_0^2$ contains a renormalized composite operator [8] $[\phi^2]$, and K is a standard leptonic factor. We have chosen to include a factor of the analog of the electromagnetic coupling in the operator. This ensures that as regards strong interactions the operator is scale invariant; the "electromagnetic" coupling e in our model gets renormalized by strong interaction effects unlike the case of the true electromagnetic coupling in QCD.

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Any exclusive component of the cross section can be obtained from $\sigma[W]$ by functional differentiation with respect to W, while, by the choice of a suitable form for $W(f)$, jet cross sections and in general any kind of binned cross section, can be obtained. The use of weighted cross sections is very compatible with the MCEG approach, since the output of a generator is a list of events with a distribution corresponding to an approximation to the fully differential cross section. An estimate of $\sigma[W]$ is obtained from a sum over the generated events weighted by $W(f)$:

$$\sigma_{\text{est}}[W] = \frac{1}{L} \sum_i W(f_i)\,, \tag{3.3}$$

where f_i is the final-state of event i, and L is the luminosity appropriate for the set of events. If the EG gives weighted events, then a factor of the event weight w_i needs to be inserted inside the summation.

4. Hard-scattering factorization

The key idea of our method is that all partons should be generated with their exact kinematics. So in this section we will obtain a factorization theorem, eq. (4.31) below, which expresses the general weighted cross section $\sigma[W]$ in terms of parton correlation functions.

While we will use the new factorization property for generating the partonic content of DIS events, we will also need the usual factorization theorem to determine values of the inclusive cross section:

$$\sigma[1] = K \int_x^1 \frac{d\xi}{\xi}\, C\!\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, g(\mu)\right) f(\xi;\mu) + \text{power-suppressed correction}\,, \tag{4.1}$$

where C is the hard scattering coefficient and f is the ordinary parton density in the target. We choose to define the parton density in the $\overline{\text{MS}}$ scheme.[6] The coefficient C is perturbatively calculable if μ is chosen to be of order Q. Notice that this theorem only applies after an unweighted sum over final states, so the argument of σ is a weight function whose value is unity for all states.

The normalizations are chosen so that the lowest order value of the hard-scattering coefficient, associated with the handbag diagram with neglect of hadronization, reproduces the parton model:

$$C\!\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, g\right) = e^2 \xi \delta(\xi - x) + O(e^2 g^2)\,, \tag{4.2}$$

so that

$$\sigma[1] = K\, e^2\, f(x;\mu)\left[1 + O\left(g^2(Q)\right)\right]. \tag{4.3}$$

4.1 Regions

The regions that give leading power contributions for a DIS process are obtained by standard arguments [9] and are illustrated in figure 4: Momenta in the lower subgraph F are

[6] As usual, the dependence of f on the renormalization scale μ includes the effect of the running coupling and mass $g(\mu)$ and $m(\mu)$, and is governed by a DGLAP equation.

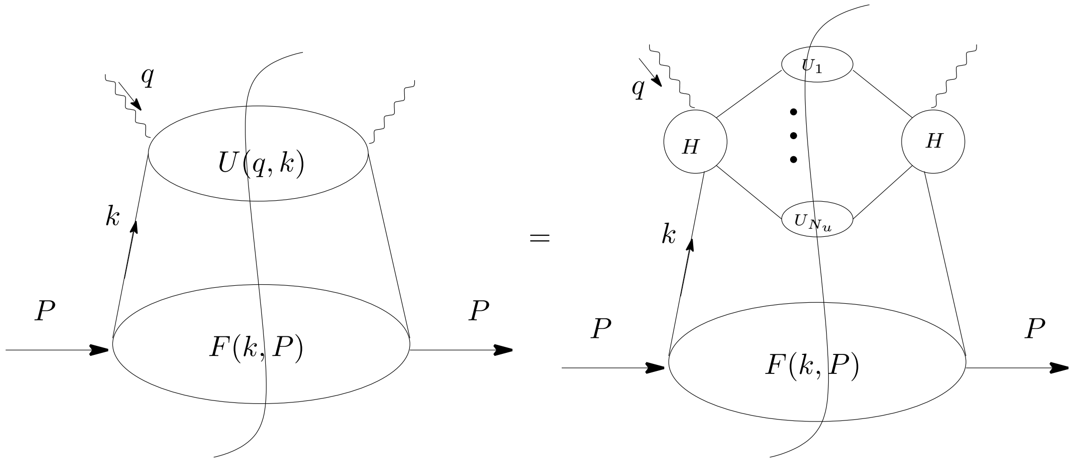

Figure 4: Decomposition of a graph for DIS corresponding to a generic leading region.

approximately in the proton direction, and momenta in the upper subgraph U are grouped around one or more different directions. The hard subgraph H has internal momenta that are all off-shell by order Q^2. If momenta could be cleanly separated into different regions, a sum over all graphs and over all regions for each graph would give quite directly the factorization formula, eq. (4.31) below, which corresponds to the factorized structure in figure 4. Moreover interpreting the blobs in figure 4 as complete amplitudes, not just as sums over Feynman graphs, strongly indicates what factorization should mean non-perturbatively in the full theory, even if we are not yet able to make a real proof beyond perturbation theory.

However, there are momentum configurations that interpolate between core parts of the different regions and these give well-known logarithmic enhancements, evolution of parton densities, etc. So an unambiguous and clean separation between different regions of momentum does not exist. All but the simplest graphs for the cross section (or structure function) have several leading regions. We will analyze them region by region, obtaining an expression for each graph Γ as a sum of terms, one for each region R:

$$\Gamma = \sum_R C_R(\Gamma) + \text{power-suppressed correction}\,, \tag{4.4}$$

with the application of subtractions to compensate double counting between regions. The principles of the argument will be exactly those of our previous work [2] for e^+e^- annihilation. The changes will be those to accommodate an initial-state hadron and an algorithm for backward showering for the initial state.

We will find that subtractions only need to be applied inside the hard scattering, so that summing over graphs and regions will give a factorized form for the weighted cross section. Schematically

$$\sigma[W] = K \sum_f \sum_{\substack{\text{graphs }\Gamma\\ \text{regions }R}} \hat{H}_R \prod_{j=1}^{N_u} U_{Rj}\, F_R\, W(f) + \text{power-suppressed correction}\,. \tag{4.5}$$

We use a subscript R to denote the values of quantities that are appropriate to a particular region of a particular graph. This will allow us to use the unsubscripted values for the corresponding quantities H, and F, etc, summed over the relevant subgraphs, that appear in the final factorization formula. Thus $\hat{H}_R$ is contribution to the subtracted hard scattering factor, from the H subgraph in figure 4. Similarly F_R and U_{Rj} correspond to the target and final-state "jet" factors. We have made explicit the sum/integral over final states, so that we can insert a non-factorized weight $W(f)$ to analyze the final-state structure. We will see that the sum over graphs can be performed separately for $\hat{H}_R$, F_R and U_{Rj}, and this will lead to the actual factorization formula, eq. (4.31) below.

4.2 Factorized approximation for one region

We now construct a factorized approximation for a general leading region R of a general graph Γ for DIS. The approximation must be compatible with the structure of a MCEG, as described in section 2, which means that it must have a factorized form and must preserve exact parton momenta for the final state. The accuracy of the approximation will degrade as momenta move away from the core of the region, and the full factorization theorem will contain subtractions in the definition of the hard scattering coefficient that eliminate double counting when we combine the contributions of different regions. Then the final answer will be uniformly accurate in all the different regions and in the interpolating regions.

It is useful to define a systematic notation for the momenta in figure 4:

- N_u is the number of the final-state partons connecting the hard scattering and the upper "jet" subgraphs in $U(q,\, k,\, m)$.
- n_j is the number of final-state particles arising from parton j in subgraph U_j.
- The momenta of these particles are $p_{j,1},\, p_{j,2},\, \ldots,\, p_{j,n_j}$.
- $\underline{p}_j$, $j = 1, \ldots, N_u$, with an underline, denotes the *collection* of the final-state particle momenta in the jet subgraph U_j, i.e., $\underline{p}_j = (p_{j,1},\, p_{j,2}, \ldots,\, p_{j,n_j})$.
- $l_j \equiv \sum_{i=1}^{n_j} p_{j,i}$ denotes the momentum of parton j, and M_j denotes its invariant mass.
- $\underline{l}_u = (l_1,\, l_2, \ldots,\, l_{N_u})$ denotes the *collection* of the final-state parton momenta.
- $\underline{M} = (M_1,\, M_2,\, \ldots,\, M_{N_u})$ denotes the *collection* of the final-state parton invariant masses.
- $\underline{\underline{p}}_u = (\underline{p}_1,\, \underline{p}_2, \ldots,\, \underline{p}_{N_u})$ denotes the *collection* of *all* the final-state particles in the upper part of the graph, $U(q,k,m)$.
- Similarly we use index $j = 0$ for the target subgraph F, *i.e.*
  - n_0 is the number of final-state particles in the target subgraph F.
  - $\underline{p}_0$ denotes the *collection* of the final-state particle momenta in F.

We introduce the following notations for the final-state phase space:

$$
\begin{aligned}
dL\Big(\underline{\underline{p}}_u; q+k; m_{\rm ph}\Big) &= \prod_{j=1}^{N_u}\prod_{i=1}^{n_j} \frac{d^5\vec{p}_{j,i}}{(2\pi)^5 2\sqrt{\vec{p}_{j,i}^{\,2}+m_{\rm ph}^2}} (2\pi)^6 \delta^{(6)}\left(\sum_{j,i} p_{j,i} - q - k\right), \\
dL(\underline{p}_j; l_j; m_{\rm ph}) &= \prod_{i=1}^{n_j} \frac{d^5\vec{p}_{j,i}}{(2\pi)^5 2\sqrt{\vec{p}_{j,i}^{\,2}+m_{\rm ph}^2}} (2\pi)^6 \delta^{(6)}\left(\sum_{i} p_{j,i} - l_j\right), \\
dL\left(\underline{l}_u; q+k; \underline{M}\right) &= \prod_{j=1}^{N_u} \frac{d^5\vec{l}_j}{(2\pi)^5 2\sqrt{\vec{l}_j^{\,2}+M_j^2}} (2\pi)^6 \delta^{(6)}\left(\sum_{j} l_j - q - k\right).
\end{aligned}
\tag{4.6}
$$

Here $m_{\rm ph}$ is the physical mass of the true final-state particles. As pointed out in [2], the integration over the final-state phase space is factorizable:

$$
\begin{aligned}
dL\left(\underline{\underline{p}}_u; q+k; m_{\rm ph}\right) &= \prod_{j=1}^{N_u}\prod_{i=1}^{n_j} \frac{d^5\vec{p}_{j,i}}{(2\pi)^5 2E_{j,i}} (2\pi)^6 \delta^{(6)}(\textstyle\sum_{j,i} p_{j,i} - q - k) \\
&= \left(\prod_{j=1}^{N_u} \int \frac{dM_j^2}{2\pi} \int \frac{d^5\vec{l}_j}{(2\pi)^5 2\sqrt{\vec{l}_j^{\,2}+M_j^2}}\right) (2\pi)^6 \delta^{(6)}(\textstyle\sum_j l_j - q - k) \times \\
&\quad \times \prod_{j=1}^{N_u} \left[\prod_{i=1}^{n_j} \frac{d^5\vec{p}_{j,i}}{(2\pi)^5 2E_{j,i}} (2\pi)^6 \delta^{(6)}(\textstyle\sum_i p_{j,i} - l_j)\right] \\
&= \left(\prod_{j=1}^{N_u} \int \frac{dM_j^2}{2\pi}\right) \int dL(\underline{l}_u; q+k; \underline{M}) \prod_{j=1}^{N_u} dL\Big(\underline{p}_j; l_j; m_{\rm ph}\Big).
\end{aligned}
\tag{4.7}
$$

With this notation, we now write a decomposition for a graph Γ and a particular region R of the form in figure 4:

$$
\Gamma[W] = \int \frac{d^6k}{(2\pi)^6} dU_R\left(\underline{\underline{p}}_u, k\right) dF_R\left(\underline{p}_0, k, P\right) W(f) \tag{4.8}
$$

with

$$
\begin{aligned}
dU_R\left(\underline{\underline{p}}_u, k\right) &= dL\left(\underline{\underline{p}}_u; q+k; m_{\rm ph}\right) \prod_{j=1}^{N_u} U_{Rj}\left(\underline{p}_j\right) H_R\left(\underline{l}_u,\, k;\, q;\, m\right), &(4.9)\\
dF_R\left(\underline{p}_0, k, P\right) &= dL\left(\underline{p}_0; P-k; m_{\rm ph}\right) F_R\left(\underline{p}_0, P\right). &(4.10)
\end{aligned}
$$

Note that at this point we have not changed or approximated the graph in any way; we have simply written the graph in a form suitable for analyzing it in region R.

In [2], integrated jet factors were defined corresponding to sums of each of U_{Rj} over graphs and final states. These serve as normalization factors in the MC algorithm for showering. Because of the different organization of the algorithm in this paper, our definitions will be slightly different. First we define an integrated jet factor as

$$
J_I(l^2; \mu) = \int d^6y \langle 0|\phi(y)\phi(0)|0\rangle e^{il\cdot y}. \tag{4.11}
$$

For consistency with our naming for the target-related factor, we can also call this a "vacuum parton correlation function" or a "final-state parton correlation function".

Similarly, for the target factor, we define a parton correlation function (pCf) $\Phi(k, P)$ as the full non-perturbative quantity that corresponds to integrating F over all final states and then summing over graphs:

$$\begin{aligned}\Phi(k,P) &= \sum_{\substack{\text{graphs}\\ \text{final states}}} dL\left(\underline{p}_0; P-k; m_{\rm ph}\right) F_R\left(\underline{p}_0, P\right) \\ &= \int d^6y \langle P|\phi(y)\phi(0)|P\rangle e^{-ik\cdot y}\,. \end{aligned} \tag{4.12}$$

The pCf is necessarily nonnegative:

$$\begin{aligned}\Phi(k,P) &= \sum_Y \int d^6y \langle P|\phi(0)|Y\rangle e^{i(P-P_Y)\cdot y}\langle Y|\phi(0)|P\rangle e^{-ik\cdot y} \\ &= \sum_Y (2\pi)^6 \delta(k+P_Y-P)\,|\langle P|\phi(0)|Y\rangle|^2 \\ &\geq 0\,, \end{aligned} \tag{4.13}$$

as is the integrated jet factor J_I.

We could now formulate factorization with an off-shell hard scattering. But on-shell and massless amplitudes are easier to compute. So we will now define an approximation appropriate to region R in terms of a hard scattering defined with massless external partons. We will denote the approximation by an operation T_R:

$$\begin{aligned} T_R\Gamma[W] \equiv & \int \frac{d\hat{k}^+}{2\pi} \int \frac{dk^-\, d^4\vec{k}_T}{(2\pi)^5} \int dL(\underline{p}_0; P-k, m_{\rm ph}) F_R(\underline{p}_0, P) \times \\ & \times \int dL(\hat{\underline{l}}_u; \hat{k}+q; M=0) H_R(\hat{\underline{l}}_u; \hat{k}, m\to 0) \times \\ & \times \prod_j \left[\int \frac{dM_j^2}{2\pi} \int dL(\underline{p}_j; l_j; m_{\rm ph})\, \Theta(M_j^2/\mu_J^2)\, U_{Rj}(\underline{p}_j) \right] W(f)\,. \end{aligned} \tag{4.14}$$

In this formula we have used a projection of the true parton momenta k, l_1, ..., l_{N_u} onto massless momenta $\hat{k}$, $\hat{l}_1$, ..., $\hat{l}_{N_u}$, with zero transverse momentum for k; we will define such a projection in section 4.3. We have defined the hard scattering to be computed with massless external momenta and with zero mass parameter for the internal lines. However, the target factor F_R and the final-state jet factors U_{R_j} are computed with the exact parton momentum k and $\underline{l}_u$, not the approximation.

As in [2], there is a cut-off function $\Theta(M_j^2/\mu_J^2)$ that restricts the final-state parton masses to be below some factorization scale[7] μ_J, which should be of order Q. An ordinary θ-function $\Theta(M_j^2/\mu_J^2) = \theta(1 - M_j^2/\mu_J^2)$ would be suitable, although a smooth cut-off might

[7] Conventionally, the term "factorization scale" applies to a quantity analogous to μ_J but for the treatment of initial-state parton radiation in the conventional factorization formalism. But the same principles apply in fragmentation functions and the like in the final state, so we will use the same name. In [2], the symbols μ_F and μ_R were used to denote the quantities that are μ_J and μ in this paper.

be better for implementation in MCEGs. After we construct a factorization formula, this cutoff function will also appear in the subtractions in the hard scattering. Since the cutoff function does not appear in the definition of the cross section itself, there is a kind of generalized renormalization-group invariance: Changes in the value of μ_J and in the functional form of Θ will leave the factorized approximation to the cross section unchanged, up to power-law corrections, when all orders of perturbation theory are included.

A similar cut off function could be applied to the initial state, but the coupling of the kinematics of initial and final state showers (through energy-momentum conservation) combined with our use of exact parton kinematics from the beginning for the struck parton's momentum k, removes the need to do so.

Changing integration variables to the exact parton momenta gives

$$
\begin{aligned}
T_R\Gamma[W] = \prod_j \int \frac{d^6 l_j}{(2\pi)^6} \int dL(\underline{p}_0; P-k, m_{\mathrm{ph}}) F_R(\underline{p}_0, k, P) H_R(\hat{\underline{l}}_u; \hat{k}, m \to 0) \Delta_{N_u}(\underline{l}_u, k) \times \\
\times \prod_j \left[\int dL(\underline{p}_j; l_j; m_{\mathrm{ph}}) \Theta\!\left(\frac{M_j^2}{\mu_J^2}\right) U_{Rj}(\underline{p}_j) \right] W(f)\,, \qquad (4.15)
\end{aligned}
$$

with $k = \sum_j l_j - q$. The quantity Δ_{N_u} is the jacobian of the transformation between the exact momenta and the massless approximate momenta — see eq. (4.22) below for its value.

In a region where the final-state lines of the subgraphs U_{Rj} do in fact form jets and where $|k^2|, \vec{k}_T^2 \ll Q^2$, $T_R\Gamma$ provides a good approximation to the original graph Γ, up to power-law corrections (and the jacobian approaches unity).

Now when we treated e^+e^- annihilation in ref. [2] we found a convenient and simple factorized approximation in terms of the massless momenta. It allowed the MC generation of massless parton configurations for the hard scattering, with probabilities independent of the showering. But in DIS, the coupling of the kinematics of final and initial-state partons, as given in section 2, prevents us from having a correspondingly simple decoupling of showering from the generation of massless partons for the hard scattering. So it is not so clearly advantageous which set of variables to use here. It follows that there is also a choice of which formula for $T_R\Gamma$ is defined to contain the jacobian Δ_{N_u}. Any change here is compensated for in physical cross sections by the subtractions in higher-order corrections to the hard scattering, so this is primarily a matter of convenience; the jacobian is unity in the collinear region. Our choice probably simplifies the logic of deriving subtractions for hard scattering, since they most naturally involve massless approximations at certain points. See section 4.7 below for further details.

4.3 Definition of projection onto massless momenta and its jacobian

In this section we give one possible definition of the projection onto massless parton momenta: $\hat{k} \equiv P_{N_u}(k)$, $\hat{\underline{l}}_u \equiv P_{N_u}(\underline{l}_u)$. We require it to obey

$$\hat{k}^- = \hat{k}_T = 0\,; \qquad (4.16)$$

$$\hat{l}_j^2 = 0\,, \qquad (j = 1, \dots, N_u)\,. \qquad (4.17)$$

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We also require energy-momentum conservation, so that $\sum_j \hat{l}_j = q + \hat{k}$. This gives the following constraints;

$$
\begin{aligned}
-k^- + \sum_j l_j^- &= \sum_j \hat{l}_j^- = q^- = \frac{Q^2}{2xP^+}\,; \\
-k_T + \sum_j l_{jT} &= \sum_j \hat{l}_{jT} = \vec{q}_T = 0\,; \\
-k^+ + \sum_j l_j^+ &= -\hat{k}^+ + \sum_j \hat{l}_j^+ = q^+ = -xP^+\,.
\end{aligned}
\tag{4.18}
$$

In addition, this projection should have a smooth limit when some or any of the external momenta go to zero. Any projection would be a valid choice if it satisfies the above constraints and $P_{N_u}[P_{N_u}(l,\, k)] = P_{N_u}(l,\, k)$. It is also desirable that the formulas be invariant under boosts in the z direction.

Our chosen definition is:

$$
\begin{aligned}
&\begin{cases}
\hat{k}_T = 0\,, \\
\hat{l}_{jT} = l_{jT} - \dfrac{k_T}{N_u}\,,
\end{cases} \\
&\begin{cases}
\hat{k}^- = 0\,, \\
\hat{l}_j^- = l_j^- - \dfrac{k^-}{N_u}\,,
\end{cases} \\
&\begin{cases}
\hat{l}_j^+ = \dfrac{\hat{l}_{jT}^2}{2\hat{l}_j^-} = \dfrac{(l_{jT} - k_T/N_u)^2}{2(l_j^- - k^-/N_u)}\,, \\
\hat{k}^+ = \displaystyle\sum_{j=1}^{N_u} \hat{l}_j^+ - q^+ = \sum_{j=1}^{N_u} \frac{(l_{jT} - k_T/N_u)^2}{2(l_j^- - k^-/N_u)} - q^+\,.
\end{cases}
\end{aligned}
\tag{4.19}
$$

Given the values of k^-, k_T and of the M_js, which parameterize the deviation of the true parton momenta from the projected momenta, we find the inverse transformation:

$$
\begin{aligned}
l_{jT} &= \hat{l}_{jT} + \frac{k_T}{N_u}\,, & l_j^- &= \hat{l}_j^- + \frac{k^-}{N_u}\,, \\
l_j^+ &= \frac{l_{jT}^2 + M_j^2}{2l_j^-}\,, & k^+ &= \sum_j l_j^+ - q^+\,.
\end{aligned}
\tag{4.20}
$$

The kinematics of DIS requires that $k^2 < 0$, so either a rejection or a good selection method is needed to eliminate positive k^2 events.

Notice that, although DIS kinematics require that $q_T = 0$, our choice of projection applies to $q_T \neq 0$ without any modification.

We need the jacobian of the transformation between the exact and the approximated parton momenta. We obtain it by expressing $dL(\underline{l})$ in terms of $-$ and T components of momentum, and then using the fact that at fixed k^- and $\vec{k}_T$, the transformation from l_j^-

and l_{jT} is a simple shift. Thus

$$\begin{aligned}
dk^+ dL(\underline{l}_u; q+k; \underline{M}) &= dk^+ \prod_{j=1}^{N_u-1} \int \frac{dl_j^- d^4 l_{jT}}{(2\pi)^5 2 l_j^-} \frac{1}{2 l_{N_U}^-} 2\pi\, \delta\left(\sum_j l_j^+ - q^+ - k^+ \right) \\
&= dk^+ \prod_{j=1}^{N_u-1} \int \frac{d\hat{l}_j^- d^4 \hat{l}_{jT}}{(2\pi)^5 2 l_j^-} \frac{1}{2 l_{N_U}^-} 2\pi\, \delta\left(\sum_j l_j^+ - q^+ - k^+ \right) \\
&= d\hat{k}^+ \prod_{j=1}^{N_u-1} \int \frac{d\hat{l}_j^- d^4 \hat{l}_{jT}}{(2\pi)^5 2 \hat{l}_j^-} \frac{1}{2 \hat{l}_{N_U}^-} 2\pi\, \delta\left(\sum_j \hat{l}_j^+ - q^+ - \hat{k}^+ \right) \left(\prod_{j=1}^{N_u} \frac{\hat{l}_j^-}{l_j^-} \right) \\
&= d\hat{k}^+ dL(\underline{\hat{l}}_u; q+\hat{k}, 0) \left(\prod_{j=1}^{N_u} \frac{\hat{l}_j^-}{l_j^-} \right). \qquad (4.21)
\end{aligned}$$

Hence the jacobian is

$$\Delta_{N_u}(\underline{l}_u, k) \equiv \frac{d\hat{k}^+ dL(\underline{\hat{l}}_u; q+\hat{k}, 0)}{dk^+ dL(\underline{l}_u; q+k; \underline{M})} = \prod_{j=1}^{N_u} \frac{l_j^-}{\hat{l}_j^-} = \prod_{j=1}^{N_u} \frac{l_j^-}{l_j^- - k^-/N_u}. \qquad (4.22)$$

In the collinear limit for the initial-state, where k^- and $\vec{k}_T$ approach zero, the jacobian approaches unity.

4.4 Construction and proof of factorization

The proof of factorization proceeds as in [2], with very few changes. We first define the contribution $C_R(\Gamma)$ associated with a leading region R of a graph Γ, using the operation T_R and some subtractions. Then we show that with this definition, the sum over R of $C_R(\Gamma)$ gives a good approximation to the graph everywhere, i.e., the remainder in eq. (4.4) is power suppressed. Then we sum over graphs for the cross section. Using the structure of $T_R\Gamma$ and of $C_R(\Gamma)$ we show that this gives a factorization. For the most part it will be sufficient to indicate the differences from [2].

To allow factorization to be a practical tool for calculations, we need the hard-scattering factor $\hat{H}$ to be perturbatively calculable, and we need to construct an algorithm for showering with perturbatively calculable evolution. Perturbative calculability of $\hat{H}$ follows just as [2], since we have the same subtractive structure; if we set the renormalization and factorization scales μ and μ_J to be of order Q, then the integrals for $\hat{H}$ are dominated by hard momenta on a scale Q. We will construct a suitable showering algorithm in section 6. Before we do this, we will, in section 5, analyze the properties of the parton correlation functions; this analysis re-uses the methods for factorization of the cross section.

Term for a given region. As in [2], each leading region is labeled by a particular pinch-singular surface (PSS) [9] in a massless theory, and, as illustrated in figure 4, a leading region R of a graph Γ is characterized by a decomposition into subgraphs: a hard subgraph, one initial-state "jet subgraph", and one or more final state "jet subgraphs". We defined an approximation T_R appropriate to the region in eq. (4.15); its action can be symbolized as converting

$$\Gamma = F_R \times H_R \times U_{R1} \times U_{R2} \times \cdots \times U_{RN_u} \tag{4.23}$$

to

$$T_R\Gamma = F_R \times (T_R H_R) \times U_{R1} \times U_{R2} \times \cdots \times U_{RN_u} \times \Delta_{N_u} \,. \tag{4.24}$$

For the representation of the graph as a sum over contributions from regions, we will define the contribution of region R by applying T_R to the graph, but only after subtracting the contributions from smaller regions, to avoid double counting. As in [2], we slightly modify this, to include a "veto factor" V_R:

$$C_R(\Gamma) = V_R T_R \left(\Gamma - \sum_{R'<R} C_{R'}\Gamma \right) . \tag{4.25}$$

These equations give a recursive definition of the contribution $C_R(\Gamma)$ for region R, which starts from the smallest regions. For a minimal region, $R_{\min}$ i.e., one with no subregions, the definition is simply $C_{R_{\min}} = T_{R_{\min}}\Gamma$.

Veto factor. The veto factor arises because [2] the integrals in the cross section include regions in the hard scattering where partons are collinear. Now the subtractions in the hard scattering imply that the corresponding contributions to the cross section are power suppressed. Thus in infra-red- and collinear-safe cross sections, the veto factor is unnecessary. But the contributions are not completely well behaved, so we define a veto factor, to eliminate small regions around the singular surfaces for the massless hard scattering, as follows:

$$V_R = \prod_S \left[V\left(\frac{\tilde{M}_S^2}{m^2}\right) V\left(-\frac{\tilde{k}_S^2}{m^2}\right) \right] . \tag{4.26}$$

The product is over all subsets S of the (massless) final-state parton lines of the hard scattering. In this formula $\tilde{M}_S$ is defined to be the invariant mass of these lines, i.e.,

$$\tilde{M}_S^2 = \left(\sum_{j\in S} \hat{l}_j \right)^2 \geq 0 \,, \tag{4.27}$$

while $\tilde{k}_S^2$ is defined to be the momentum transfer with respect to the incoming parton line:

$$\tilde{k}_S^2 = \left(\hat{k} - \sum_{j\in S} \hat{l}_j \right)^2 \leq 0 \,. \tag{4.28}$$

The elementary veto factor is defined as

$$V\left(\frac{M^2}{m^2}\right) = \theta(M-m) \,. \tag{4.29}$$

Thus the veto factor V_R for region R eliminates a small neighborhood of the initial-state and final-state collinear singularities associated with the massless partons in the hard scattering. The exact definition is irrelevant.

We will show an example of the initial-state veto factor in section 7.

Factorization. Now that appropriate definitions have been made, the proof of factorization given in [2] applies to the case of DIS. When we construct $C_R(\Gamma)$, the subtractions are for smaller regions. As in [2], a smaller region implies larger collinear subgraphs and therefore a strictly smaller hard subgraph. Therefore the subtractions are applied only inside the hard subgraph:

$$C_R(\Gamma) = F_R \times V_R T_R \left(H_R - \sum_{R'<R} C_{R'} H_R \right) \times U_{R1} \times \cdots \times U_{RN_u} \times \Delta_{N_u} \,. \tag{4.30}$$

This contrasts with the case of gauge theories, where the presence of leading effects due to soft gluons gives a more complicated situation, which needs disentangling with the aid of Ward identities.

The factorization formula is now obtained by summing over all regions and graphs. The correspondence between regions and subgraphs converts the sum to independent sums over graphs for F_R, H_R, N_u and the U_{Rj}s. Hence

$$\sigma[W] = K \sum \int \hat{H}_{N_u} \Delta_{N_u} F(\underline{p}_0, P) \prod_{j=1}^{N_u} \left[J(\underline{p}_j)\, \Theta\!\left(\frac{M_j^2}{\mu_J^2} \right) \right] W(f), \tag{4.31}$$

up to power-suppressed corrections, as usual. The sum and integral represent a sum and integral over final states and then a sum over decompositions of the partonic structure and over the assignment of final state particles to the F and J factors. Explicitly, the factors are as follows:

1. The hard factor $\hat{H}_{N_u}$ is obtained by summing the hard scattering factor in eq. (4.30) over all graphs for H_R:

$$\hat{H}_{N_u}(\hat{k}, \underline{\hat{l}_u}; Q, \mu_J, \mu, g(\mu)) = \sum_{\text{graphs}} V\, T \left(H_R - \sum_{R'<R} C_{R'} H_R \right), \tag{4.32}$$

with a given number of final-state partons. The relevant graphs are for an amputated amplitude times complex conjugate amplitude with N_u final-state parton lines, one incoming parton line, and one virtual photon. The unsubscripted T denotes the operation of projecting the external partons onto massless momenta and setting to zero the mass m on internal lines, while V without a subscript is the corresponding veto factor. The proof in [2] shows that $\hat{H}$ has no collinear divergences.

2. The initial-state factor F is the sum over graphs of F_R for a given final-state. Thus we have an operator formula

$$F(\underline{p}_0, P; \mu) = \left| \langle P | \phi(0) | \underline{p}_0 \rangle \right|^2 . \tag{4.33}$$

The integral of F over all final states with a given value of k ($k = P - \sum_i p_{0,i}$) gives the pCf

$$\sum_{n_0=1}^{\infty} \int dL(\underline{p}_0; P-k, m_{\text{ph}})\, F(\underline{p}_0, P) = \Phi(k, P) \,. \tag{4.34}$$

3. Similarly we have a final-state factor for each final-state parton, which we define exactly as in [2]:

$$J(\underline{p}_j;\mu) = \left|\langle 0|\phi(0)|\underline{p}_j\rangle\right|^2 . \tag{4.35}$$

It obeys a sum rule

$$\sum_{n_j} \int dL(\underline{p}_j; l_j, m_{\rm ph}) J(\underline{p}_j) = J_I(l_j^2;\mu) . \tag{4.36}$$

We now reorganize the factors in eq. (4.31) to be suitable for a MCEG. We assume that we have generated values of the usual x and Q from standard factorization (4.1), these variables determining the lepton kinematics. The problem is now to find probability densities for the internal variables so that the generated events give correct weighted exclusive cross sections. So we first normalize the factorization formula for $\sigma[W]$ by dividing by the inclusive cross section $\sigma[1]$. Then we choose as independent variables the massless variables, $\hat{k}$ and $\hat{l}_j$, together with the variables k^-, $\vec{k}_T$, and M_j that characterize the deviation from exact collinearity. We write a factor that can be used to generate a hard scattering in terms of the massless projected momenta independently of the showering. Then we write a factor that gives an approximately normalized[8] probability density for k^-, $\vec{k}_T$, and factors that give normalized probability densities in M_j, and for the final states resulting from each final-state parton. Finally we insert a factor that expresses the deviation of the true formula from an exactly factored form. We find

$$\begin{aligned} \frac{\sigma[W]}{\sigma[1]} = & \sum_{N_u=1}^{\infty} \int \frac{d\hat{k}^+\, dk^-\, d^4\vec{k}_T}{(2\pi)^6} \int_{\underline{\hat{l}}_u} \frac{d\hat{\sigma}_{N_u}(Q,\mu_J,\mu)}{\sigma[1]} f\left(\frac{\hat{k}^+}{P^+};\mu\right) \frac{\Phi((\hat{k}^+,k^-,\vec{k}_T),P;\mu)}{f(\hat{k}^+/P^+;\mu)} \times \\ & \times \prod_{j=1}^{N_u} \left[\int_0^\infty \frac{dM_j^2}{2\pi} \, \frac{J_I(M_j^2;\mu)\,\Theta(M_J^2/\mu_J^2)}{Z(\mu_J/\mu, m/\mu)} \right] \times \\ & \times \left[\sum_{n_0} \int \frac{dL(\underline{p}_0; P-k; m_{\rm ph})\ F(\underline{p}_0, P;\mu)}{\Phi(k,P;\mu)} \right] \prod_{j=1}^{N_u} \left[\sum_{n_j} \int dD(\underline{p}_j; l_j;\mu) \right] \times \\ & \times \frac{\Phi(k,P;\mu)}{\Phi((\hat{k}^+,k^-,\vec{k}_T),P;\mu)} W(f) . \end{aligned} \tag{4.37}$$

Here f is the ordinary renormalized parton density, discussed in section 5.2 below. Notice that we have defined Φ by an operator matrix element in the exact theory, in which the particle mass is nonzero. Thus the integral over k is free of the collinear divergences that arise in conventional calculations in a massless theory.[9] In addition, k is the exact quark momentum, so that the upper limits on its components are set by the kinematics of the process, and there are no ultra-violet problems. Thus the integral is convergent.

[8] In a sense to be discussed below.

[9] We only make a massless approximation in the fully subtracted hard-scattering coefficients, where the approximation is valid.

In the first line we use what we will call the differential hard-scattering cross section defined by

$$d\hat{\sigma}_{N_u} = K\, dL(\hat{\underline{l}}_u; q + \hat{k}, 0)\, \hat{H}_{N_u}(Q, \mu_J, \mu)\, Z\left(\frac{\mu_J}{\mu}, \frac{m}{\mu}\right)^{N_u}. \tag{4.38}$$

As in [2], we have inserted a factor Z for each outgoing parton. This provides a kind of analog to the propagator residue factors in the LSZ reduction formula, and is defined by

$$\begin{aligned} Z\left(\frac{\mu_J}{\mu}, \frac{m}{\mu}\right) &= \int_0^\infty \frac{dM_j^2}{2\pi}\, \Theta\left(\frac{M_j^2}{\mu_J^2}\right) J_I(M_j^2; \mu) \\ &= \int_0^\infty \frac{dM_j^2}{2\pi} \sum_{n_j} \int dL(\underline{p}_j; l_j) \Theta\left(\frac{M_j^2}{\mu_J^2}\right) J(\underline{p}_j; \mu)\,. \end{aligned} \tag{4.39}$$

Observe that [2] since the integral of J_I over all M_j diverges at large M_j, we have to use a cutoff, the factorization scale μ_J, to define a suitable finite quantity. We have implemented the cutoff by the function $\Theta(M_j^2/\mu_J^2)$ rather than by the upper limit of integration, to allow for the possibility of a smooth cutoff.

But there appears to be no unambiguously appropriate analog of the factor Z for the incoming parton. Instead, on the first line of eq. (4.37), we have multiplied $d\hat{\sigma}$ by the ordinary parton density $f(\hat{k}^+/P^+; \mu)$ taken at the approximated parton momentum. This is like the parton density times hard-scattering in the conventional formalism, and we can use the numerical values of $d\hat{\sigma} f$ for generating the kinematic variables $\hat{k}^+$ and $\hat{l}_j$. However, as we have already argued, the value of $\hat{k}^+$ does not agree with the correct value of true parton momentum. Since the correct value cannot be known algorithmically until the full parton kinematics are known, we insert a correction factor on the last line of eq. (4.37) to adjust the full cross section to its correct value and we will implement the correction factor as a reweighting.

To a first approximation, the incoming parton for the hard scattering has low transverse momentum and virtuality, and then k^+ is close to $\hat{k}^+$. So $d\hat{\sigma}\, f/\sigma[1]$ is approximately a normalized distribution in $\hat{k}^+$ and the outgoing parton momenta. Hence the reweighting corrections are mild.

The remaining factor on the first line, Φ/f, is therefore also approximately a normalized probability density in k^- and $\vec{k}_T$, with upper limits determined kinematically. The density f, conventionally called an "integrated parton density", is therefore approximately an integral over Φ. However, because of complications from ultra-violet behavior these statements are not exact; the detailed relations between Φ and f will be explored in section 5. Again, reweighting corrections to compensate for slightly inaccurate normalization are mild.

In the second line, the distribution in outgoing parton mass is given by the J_I/Z factors. Given our cutoff at $M_j = \mu_J$, this is a correctly normalized conditional probability density.

The normalized differential distribution of final-states in an out-going parton given the parton's exact momentum is the quantity dD defined in eq. (6.42) of [2]. In our current notation it is

$$dD(\underline{p}_j; \mu) = \frac{J(\underline{p}_j; \mu)\, dL(\underline{p}_j; l_j)}{J_I(l_j^2; \mu)}\,. \tag{4.40}$$

The corresponding normalized initial-state factor is $dL(\underline{p}_0) F/\Phi$.

The deviation from the factorized form of the probabilities is given by the ratio of the pCf at the correct incoming parton momentum to the value at the approximated (but off-shell) momentum, i.e., by the factor $\Phi(k, P)/\Phi((\hat{k}^+, k^-, \vec{k}_T), P)$. This factor approaches unity in the collinear limit. In a MCEG it can be implemented by either weighting generated events by the factor or using a suitable veto strategy. Because the weighting factor is well behaved in the collinear limit, the veto method is reasonable. The important point for the logic is that this reweighting factor is used only within the generation of the hard scattering, which in our method includes the generation of the exact off-shell momenta of the external partons of the hard scattering.

4.5 Choices of μ and μ_J

The renormalization scale μ and the factorization scale μ_J are artifacts of our method of treatment of the theory, so that the cross section does not depend on them. As usual, we exploit the dependence on these variables of factors in factorization theorems to enable perturbative calculations to be done without large logarithms. Scale dependence cancels in predictions of cross sections up to the effect of uncalculated higher order corrections.

First, the inclusive cross section is calculated from the standard factorization formula eq. (4.1) with μ of order Q; then the coefficient function is a weak-coupling expansion in $g(\mu)$ without large logarithms. The parton density at scale $\mu \sim Q$ is related to the parton density at a fixed scale Q_0 by use of the ordinary DGLAP evolution. The evolution is perturbative. Our approach is to define the parton density by $\overline{\text{MS}}$ renormalization, so only the renormalization scale μ appears here; there is no separate factorization scale.

Next in the generation of an event is the use of eq. (4.37) to give the kinematic distribution of the hard scattering. Here we also set μ and μ_J of order Q so that the hard scattering is perturbative. For consistency we must use the same values everywhere in eq. (4.37). The pCf Φ is an operator matrix element, so it depends on μ but not μ_J. Its calculation and scale-dependence will be considered in section 5.

The treatment of the distribution of parton mass J_I/Z, on the second line of eq. (4.37), is exactly as in [2]. Since μ and μ_J are the same as in the rest of the formula, they are of order Q. The calculation of $Z(\mu_J/\mu, m/\mu)$ is then perturbative, in powers of $g(\mu)$, with the mass dependence being negligible. But $J_I(M_j^2; \mu)$ will have logarithms of M_j/μ, which get large[10] when $M_j \ll Q$. Its evolution is governed by the anomalous dimension of the parton field and therefore by the same RGE as the pCf, eq. (5.2) below. Thus we use the renormalization group to express it in terms of J_I at another scale:

$$J_I(M_j^2; \mu) = \exp\left[-\int_{\mu'}^{\mu} \frac{dv}{v} \gamma(g(v)) \right] J_I(M_j^2; \mu') \,. \tag{4.41}$$

When M_j is in a perturbative region, we set μ' of order M_j, so that $J_I(M_j^2; \mu')$ can be computed perturbatively in powers of $g(\mu')$. When M_j is too small, we set μ' to a fixed scale Q_0 and use a non-perturbative model (fit to data). The model is constrained by a sum rule that J_I integrates to the perturbatively calculable quantity Z.

[10]Note that as a simple operator matrix element J_I has no μ_J dependence.

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In the third line of eq. (4.37), we have factors that give the distribution of hadronic states given the exact parton momenta. Each factor is a phase-space differential times a ratio of two matrix elements. The matrix elements differ only by whether or not they are integrated over final states, so the numerator and denominator have the same anomalous dimension. Thus we can replace μ separately in each factor by a scale appropriate to each parton: of order M_j for a final-state parton or of order $\sqrt{|k^2|}$ for the initial-state parton. (A fixed scale would be more appropriate in the non-perturbative region of momenta.) Then we exploit factorization of the pCf to further the calculation. The change in the value of the scale μ, potentially very different in different factors, is the key feature that enables the MCEG approach to give systematically reliable estimates of cross sections.

The final factor $\Phi(k, P)/\Phi((\hat{k}^+, k^-, \vec{k}_T), P)$ is again a ratio of quantities with the same anomalous dimension, so that it is scale-independent. The values needed are obtained from the calculation of the pCf.

4.6 Collinear safety, renormalization group equations

For the factorization formula to have predictive power we need the following:

- The remainder, i.e., the difference between the factorized formula for the cross section and the true cross section, is power suppressed.
- Infrared and collinear safety of the subtracted hard scattering factors $\hat{H}$.
- Infrared safety of the integrated final-state jet factors.
- Renormalization-group-like equations for the scale dependence of the jet factors.
- Theorems for the parton correlation functions.

The power suppression of the error and the collinear and infra-red safety follow exactly as in [2], and the evolution equations for the final-state jet factors are the same. Then as explained in the previous section, section 4.5, perturbatively based calculations can be made by combining suitable choices of μ and μ_J in each factor with the evolution equations to relate the factors to their values with a common choice of scale.

4.7 Factorization schemes

We have a choice in the algorithm for generating parton configurations for the hard scattering:

(a) We can follow the structure of eq. (4.37): (i) Generate a conventional massless hard scattering. (ii) Generate values for k^-, $\vec{k}_T$, and the M_j's. (iii) Compute the exact parton momenta k and $\underline{l}_u$. (iv) Reweight by the ratio of the correct pCf to the approximated pCf. Or:

(b) We can generate the hard scattering according to a formula in terms of the parton correlation functions alone with exact parton momenta.

In both cases initial-state and final-state showering are performed conditional on the exact parton momenta, so the change in algorithm affects only the algorithm for the hard scattering, not the rest of the MCEG or the probabilities. For the second case, we can exhibit the necessary probability densities by rewriting eq. (4.37) in a form closer to that of eq. (4.31):

$$\frac{\sigma[W]}{\sigma[1]} = \sum_{N_u=1}^{\infty} \prod_{j=1}^{N_u} \left[\int \frac{d^6 l_j}{(2\pi)^6} J_I(l_j^2)\, \Theta\!\left(\frac{l_j^2}{\mu_J^2}\right) \right] \frac{K\, \hat{H}_{N_u}\, \Phi(k,P)\, \Delta_{N_u}(\underline{l}_u, k)}{\sigma[1]} \times \tag{4.42}$$
$$\times \left[\sum_{n_0} \int \frac{dL(\underline{p}_0; P-k; m_{\rm ph}) F(\underline{p}_0, P)}{\Phi(k,P)} \right] \prod_{j=1}^{N_u} \left[\sum_{n_j} \int dD(\underline{p}_j; l_j; \mu) \right] W(f)\,,$$

with $k = \sum_j l_j - q$.

Which of these algorithms will be better in practice is not yet clear. That will depend on practical experience, at least.

But clearly, if we use the second method, it would appear more natural to remove the jacobian factor Δ_{N_u} from eq. (4.42). If we did this, then consistency between the two event-generation methods requires us also to *insert* a factor of $1/\Delta_{N_u}$ in the formula (4.37) used to specify the probabilities in the first scheme. We would also need to remove the factor Δ_{N_u} from eq. (4.31). Consistency with the derivation of these equations requires a corresponding change in the definition of T_R: we would insert a factor $1/\Delta_{N_u}$ in eq. (4.14) and remove a factor Δ_{N_u} from eq. (4.15).

Since T_R changes, we would find that changes in the higher-order terms in the hard-scattering coefficients occur, as is forced by the subtractions in the recursive definition (4.25) of the contribution C_R of an individual region. This gives us two schemes, each more natural to one of the two event generation algorithms. As usual, changes in the scheme are compensated in higher-order corrections to the hard scattering, so that the physical cross sections are invariant up to uncalculated corrections of yet higher order.

5. Parton correlation functions (pCf's)

To build an EG for DIS we will need the values of the pCf $\Phi(k,P)$, to use in the hard-scattering factorization formula. At large k^2, this will involve a new factorization formula that expresses it in terms of ordinary parton densities (defined, say, in the $\overline{\rm MS}$ scheme). This is exactly analogous to the standard factorization property for the cross section, eq. (4.1). The showering of the parton, and hence the analysis of the final states inside Φ, will need a second factorization formula analogous to eq. (4.37) or (4.42) for the cross section.

The second factorization formula for the pCf concerns a quantity $\Phi[W]$ whose relation to the pCf Φ is like that of the weighted cross section $\sigma[W]$ to the unweighted DIS cross section $\sigma[1]$. To be precise, we define

$$\Phi_{[W]}(k,P;\mu) = \sum_{n_0} \int dL(\underline{p}_0; P-k; m_{\rm ph}) F(\underline{p}_0, P; \mu) W(f)\,, \tag{5.1}$$

where F is the same as in eq. (4.33). Of course, the previously defined unweighted pCf, eq. (4.34), is the weighted pCf with unit weight for all states: $\Phi(k,P) = \Phi_{[1]}(k,P)$.

This factorization formula for the weighted pCf implies an integral equation for the unweighted pCf in terms of itself, so that it is not an easy equation to solve for Φ. However the first factorization property for the pCf gives the unweighted Φ in terms of ordinary parton densities, and it therefore gives an effective method for calculating its functional form. Given this information, the second factorization is perfectly adapted to analyzing the partonic content of the final states, i.e., as a basis for the showering of an initial-state parton. In addition we need renormalization group equations so that the renormalization scale μ^2 can be chosen of order $|k^2|$ to exploit the factorization theorems perturbatively at large $|k^2|$.

The uses of the factorization theorems at large $|k^2|$ are perturbative, aided by renormalization-group transformations. But at low $|k^2|$ in an asymptotically free theory, we must resort to non-perturbative methods, modeling and simple data fitting, just as in current MCEGs.

5.1 Renormalization group

Let us first observe that the pCf (weighted or unweighted) arises from matrix elements of two renormalized fields, so it has a renormalization-group equation

$$\frac{d}{d\ln\mu}\Phi(k,p;\mu) = -\gamma(g(\mu))\Phi\,, \tag{5.2}$$

with the conventions of [8]. As usual, the total derivative applies to both the direct μ-dependence of the Green function and the μ-dependence of the running coupling and mass. This formula enables us to transform the pCf from $\mu \simeq Q$ as used in hard scattering factorization to the value $\mu^2 \simeq |k^2|$ as used when calculating the value of Φ from the first factorization or when applying the second factorization for showering.

5.2 Standard parton densities

The definition of the standard $\overline{\text{MS}}$-pdf in ϕ^3 theory is given in [3, 10]. Here we review it and some of the properties of the pdf. First there is the unrenormalized pdf:

$$f_0(x) = xP^+ \int dy^-\, e^{-ixP^+y^-}\, \langle P|\, \phi_0(0,y^-,\vec{0}_T)\, \phi_0(0)\,|P\rangle_c\,. \tag{5.3}$$

The subscript "c" on the matrix element means that we take only the component in which the fields are connected to the target particle: the vacuum expectation value is subtracted out. Implicitly, f_0 has dependence on the parameters of the theory, $g(\mu)$, $m(\mu)$ and μ, as well as on the UV regulator parameter, which we take to be the dimension n of space-time. Notice that we use bare fields in this formula, which implies that there is an ultra-violet divergence caused by that in Z_2. But, if in an attempt to obtain a UV-finite renormalized parton density, we simply replaced the bare fields by renormalized fields, that would not work because there is a further UV divergence caused by the unrestricted integral over the transverse momentum and virtuality of the external parton. So the $\overline{\text{MS}}$ pdf is defined to have all the UV divergences renormalized. It has the form[11]

$$
\begin{aligned}
f(x;\mu) &= \lim_{n\to 6}\int_1^{x^{-1}} \frac{d\eta}{\eta} K(\eta,g,n) f_0(\eta x) \\
K(\eta,g,n) &= \delta(\eta-1) + \sum_{N=1}^{\infty} (n-6)^{-N} K_N(\eta,g)\,,
\end{aligned}
\tag{5.4}
$$

where $K(\eta,g,n)$ is a series of pure poles at the physical space-time dimension $n=6$.

Notice that in the absence of these divergences, the bare parton density would be an integral over the pCf with a factor Z_2:

$$
f_0(x) = Z_2 \int \frac{dk^-\, d^4\vec{k}_T}{(2\pi)^5}\, \Phi(k,P)\,. \tag{5.5}
$$

Since we can analyze issues of UV behavior perturbatively, this equation suggests that a suitably improved analysis would relate f to an integral over Φ. Hence, the factor Φ/f in eq. (4.37) is approximately a probability distribution in k^- and $\vec{k}_T$ conditional on the value of $\hat{k}^+$. Because of the existence of higher-order corrections to the UV coefficient functions in relations between Φ and f, we must expect such statements not to be exact, however. We will give a detailed analysis in section 5.4, and merely observe here that, in a leading logarithm approximation, we can approximate renormalization by a cut-off. Thus we can approximately write the usual pdf at a large scale M as:

$$
f(x;g(M),m,\mu=M) = \int_{|k^2|,k_T^2<M^2} \frac{dk^-\, d^4\vec{k}_T}{(2\pi)^5}\, \Phi(k,P;M) + O(g(M)^2)\,. \tag{5.6}
$$

It is only in this approximate sense that the usual designation of f as an "integrated pdf" is justified.

Observe that the scale dependence of the parton density arises from two sources: the explicit cutoff on the k integral, and the anomalous dimension of the renormalized parton field in the operator definition of Φ.

5.3 Kinematics of pCf

In non-gauge theories, by Lorentz invariance, the parton correlation function $\Phi(k,P;\mu)$ depends on k only through the scalar variables k^2 and $k\cdot P$. Kinematically Φ has the same kind of variables as a DIS structure function, so it is convenient to use variables analogous to the standard x and Q^2 of DIS:

$$
\zeta \equiv \frac{k^2}{2P\cdot k}\,, \qquad R^2 \equiv -k^2\,. \tag{5.7}
$$

The kinematics in the factorization formula (4.37) for DIS correspond to the first graph in figure 4, so k is space-like. Hence, even though the pCf is defined for all values of R^2,

[11] The necessary dependence of the bare parton density on the the regulator parameter n, both directly and through the n-dependence of the bare coupling and mass, has not been indicated explicitly in this formula.

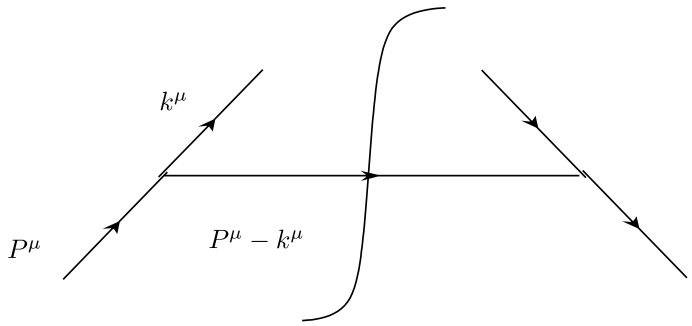

Figure 5: Tree graph for $\Phi(k, P)$ of a parton target.

it is only used when R^2 is positive. Just as in DIS, the positivity of the squared mass of the final state implies that $0 < \zeta \leq 1$, when $R \gg m$. Since the kinematics of the pCf are completely analogous to those of DIS structure functions, so are the factorization properties, essentially obtained by the replacement of currents by parton fields.

In gauge theories like QCD, the gauge-invariance of the theory requires us to insert Wilson lines in the operator definition of pCf's. Appropriate direction(s) for the Wilson lines are those compatible with a proof of factorization, a proof that would be much more complicated than in our ϕ^3 theory. If n^μ is the direction of the Wilson line, then there are two extra scalar variables: $(P \cdot n)^2/n^2$ and $k \cdot n/P \cdot n$. (Since Φ is invariant under scaling of n only variables invariant under this scaling can be used.) The extra two variables will increase the computational cost: for example in a much larger interpolation table to be used in the numerical evaluation of the pCf. In addition there will be more complications in the factorization theorems for the pCf. Note that it should be possible to obtain the dependence on one of the extra variables, probably $(P \cdot n)^2/n^2$, by a suitable generalization of the Collins-Soper equation [11].

5.4 Relations between ordinary parton density and parton correlation function

There are two kinds of relation between the ordinary pdf and the pCf Φ. The first gives a result for Φ when the parton virtuality is large. The second result gives an integral of Φ over parton momenta k.

5.4.1 pCf at large $|k^2|$

The result for large $|k^2|$ arises because, as we have already stated, the pCf is like a structure function but with different operators. When $|k^2|$ is much larger then some typical nonperturbative scale, it follows that there is a factorization theorem for the unweighted pCf of the same form as eqs. (4.1) for the DIS cross section or structure function:

$$\Phi(k, P; \mu) = \int_\zeta^1 \frac{d\xi}{\xi}\, C_\Phi\left(\frac{\zeta}{\xi}, -\frac{k^2}{\mu^2}, g(\mu)\right) f(\xi; \mu) + \text{power-suppressed correction}\,. \qquad (5.8)$$

Calculations of the coefficient C_Φ are completely analogous to calculations of the structure function F_2, except for details due to differences in the graphs. For example the lowest-order non-trivial graph for a pCf of a parton target figure 5 is

$$\Phi(k,P;\mu)_{LO} = 2\pi\,\delta((P-k)^2-m^2)\frac{g(\mu)^2}{(k^2-m^2)^2} = 2\pi\frac{\zeta}{|k^2|}\delta(\zeta-1)\frac{g(\mu)^2}{(k^2-m^2)^2}\,. \tag{5.9}$$

Notice that unlike the case of f, which is integrated over k^2, the graph without any interactions does not contribute here since we will make $|k^2|$ large. The lowest order term in the coefficient C_Φ is therefore

$$C_\Phi\left(\frac{\zeta}{\xi},R^2;\mu,g\right) = \frac{2\pi g^2}{R^6}\zeta\delta(\zeta-\xi)+O(g^4)\,. \tag{5.10}$$

Hence the lowest order estimate of the unweighted pCf is

$$\Phi(k,P;\mu) = \frac{2\pi g(\mu)^2}{|k^2|^3}f(\zeta;\mu)\,+O\left(\frac{g^4(\mu)}{|k^2|^3}\times\text{logarithms}\right), \tag{5.11}$$

where, as already mentioned, $\zeta = k^2/(2k\cdot P)$. Note, that this formula is only to be applied at large $|k^2|$. At low k^2 we cannot neglect the parton mass, and therefore the apparent singularity at $k^2=0$ in eq. (5.11) is not physical.

In the higher terms in eq. (5.11), there are, as usual in such expansions, logarithms of k^2/μ^2. Since we will use Φ in, for example, a calculation of hard scattering at scale Q, but need its value when $|k^2|$ is much less than Q^2, we must evaluate the right-hand side of eq. (5.11) after a renormalization-group transformation using eq. (5.2). Thus practical calculations will use

$$\Phi(k,P;\mu) = \frac{2\pi g^2(\sqrt{|k^2|})}{|k^2|^3}f(\zeta;\sqrt{|k^2|})\exp\left(-\int_{\sqrt{|k^2|}}^{\mu}\frac{d\mu'}{\mu'}\gamma(g^2(\mu'))\right)+O\left(\frac{g^4(|k|^2)}{|k^2|^3}\right). \tag{5.12}$$

On the right-hand side of this formula we have transformed the renormalization scale to $\sqrt{|k^2|}$. But the scale may be multiplied by a factor of order unity, which could be used to optimize for the likely size of uncalculated higher order corrections.

5.4.2 Integral of PCf

The second result for the pCf was proved (with a slight variation) by one of us in [10]. It says that if we define a quantity $\tilde{f}$ as the integral of Φ over k^- and $\vec{k}_T$, with an upper limit on $\vec{k}_T$ and virtuality:

$$\tilde{f}(x,M;g(\mu),m,\mu) = \int_{|k^2|,k_T^2<M^2}\frac{dk^-\,d^4\vec{k}_T}{(2\pi)^5}\,\Phi(k,P;g(\mu),m,\mu)\,, \tag{5.13}$$

then this has an expansion in terms of the ordinary parton density:

$$\tilde{f}(x,M;g(\mu),m,\mu) = \int_x^1 d\xi\xi^{-1}T\left(\frac{\xi}{x},\frac{M^2}{\mu^2},g(\mu)\right)f(\xi)+\text{power-suppressed correction}\,. \tag{5.14}$$

The formulation in [10] differs only in that the limit was imposed only on $\vec{k}_T$ and not on $|k^2|$ as well, but the proof is unaltered. The lowest order value of the coefficient T is $\delta(\xi/x-1)$, so that $\tilde{f} = f + O(g^2)$. From this follows the previously announced result eq. (5.6); the $\overline{\rm MS}$ pdf is approximately the integral of an unintegrated parton density.

This result gives a sum rule involving an integral over the whole non-perturbative region for the pCf, thereby providing a constraint on modeling of this region given that the ordinary pdf has already been measured.

5.5 Recursive factorization for weighted pCf in terms of pCf

In eq. (4.37) the weighted DIS cross section was expressed with an integral over exact parton momenta in terms of the unweighted pCf. With the same reasoning, there is an analogous factorization property for the (weighted) pCf. Since we will wish to use it to give conditional probability densities for the internal partons in a MCEG, we normalize the formula to the unweighted pCf:

$$\begin{aligned}
\frac{\Phi_{[W]}(k,P;\mu)}{\Phi(k,P;\mu)} = &\sum_{N_u=1}^{\infty} \int \frac{d\hat{r}^+\, dr^-\, d^4\vec{r}_T}{(2\pi)^6} \int_{\hat{\underline{l}}_u} \frac{d\hat{H}_{\Phi,N_u}(\mu,\mu'_J)}{\Phi(k,P;\mu)} f\left(\frac{\hat{r}^+}{P^+};\mu\right) \times \\
&\times \frac{\Phi((\hat{r}^+,r^-,\vec{r}_T),P;\mu)}{f(\hat{r}^+/P^+;\mu)} \times \\
&\times \prod_{j=1}^{N_u} \left[\int_0^\infty \frac{dM_j^2}{2\pi} \frac{J_I(M_j^2;\mu'_J,\mu)\,\Theta(M_j^2/\mu'^2_J)}{Z(\mu'_J/\mu,m/\mu)} \right] \times \\
&\times \left[\sum_{n_0} \int \frac{dL(\underline{p}_0;P-r;m_{\rm ph})\; F(\underline{p}_0,P;\mu)}{\Phi(r,P;\mu)} \right] \prod_{j=1}^{N_u} \left[\sum_{n_j} \int dD(\underline{p}_j;l_j;\mu) \right] \times \\
&\times \frac{\Phi(r,P;\mu)}{\Phi((\hat{r}^+,r^-,\vec{r}_T),P;\mu)} W(f)\,.
\end{aligned} \tag{5.15}$$

The main change compared with eq. (4.37) is that the differential hard scattering factor has been given a different name, $d\hat{H}_\Phi$ instead of $d\hat{\sigma}$. In addition, the symbol for the internal loop momentum has been changed from k to r, to avoid a conflict with the symbol for the external parton momentum. The differential hard scattering coefficient $d\hat{H}_\Phi$ is defined in terms of subtracted hard scattering graphs in analogy to $d\hat{\sigma}$ in eq. (4.38):

$$d\hat{H}_{\Phi,N_u}(\mu,\mu'_J) = dL(\hat{\underline{l}}_u;k+\hat{r},0)\,\hat{H}_{\Phi,N_u}(\mu,\mu'_J)\,Z\left(\frac{\mu'_J}{\mu},\frac{m}{\mu}\right)^{N_u}. \tag{5.16}$$

The approximated internal parton momenta $\hat{r}$ and $\hat{l}_j$ are defined as before, except for replacing q and k in the definitions in section 4.3 by $-k$ and r.

Notice how we have used a different symbol μ'_J for the factorization scale than we did in factorization for the cross section. When we employ eq. (5.15), we will use the fact that the anomalous dimensions of the numerator and denominator on the left-hand side are the same, so that we can replace the value $\mu \sim Q$ used in factorization for the cross section by a value $\mu \sim \sqrt{|k^2|}$ appropriate for the use of (5.15). We must also choose μ'_J of the same order, so that the hard-scattering factor in this equation is perturbatively calculable, without large logarithms of k^2/μ'^2_J or of k^2/μ^2.

It is also possible to write (5.15) in terms of unapproximated momenta, as we did for the cross section in eq. (4.42):

$$
\begin{aligned}
\frac{\Phi_{[W]}(k,P;\mu)}{\Phi(k,P;\mu)} = & \sum_{N_u=1}^{\infty} \prod_{j=1}^{N_u} \left[\int \frac{d^6 l_j}{(2\pi)^6} \, J_I(l_j^2) \right] \frac{\hat{H}_{\Phi,N_u}(\mu,\mu'{}_J)\,\Phi(r,P;\mu)\Delta_{N_u}(\underline{l}_u,r)}{\Phi(k,P;\mu)} \times \\
& \times \left[\sum_{n_0} \int \frac{dL(\underline{p}_0;P-r;m_{\rm ph})F(\underline{p}_0,P;\mu)}{\Phi(r,P;\mu)} \right] \prod_{j=1}^{N_u} \left[\sum_{n_j} \int dD(\underline{p}_j;l_j;\mu) \right] \times \\
& \times W(f)\,,
\end{aligned}
\tag{5.17}
$$

with $r = \sum_j l_j + k$.

The lowest-order value of $\hat{H}_{\Phi,1}$ is the essentially the same as the coefficient for the pCf in terms of integrated parton densities:

$$
\hat{H}_{\Phi,1}(k,\hat{r}) = \frac{2\pi g^2}{|k^2|^3} \delta\!\left(-1 + \frac{2k\cdot\hat{r}}{k^2} \right) + O(g^4)\,. \tag{5.18}
$$

The lowest-order value of the $\hat{H}_{\Phi,N_u}$ with more external outgoing partons is of order g^{2N_u}.

6. Implementation

We now put the information together to give an algorithm for a MCEG. First, the ordinary factorization formula (4.1) gives us a calculation of the inclusive cross section (and hence of the structure function) as a function of x and Q. Then (4.37) expresses the weighted cross section in terms of the pCf Exactly similar factorization formulas apply to the pCf: eqs. (5.8) and (5.15). Lowest order values for the coefficients are presented below the corresponding factorization formulas.

We now assume that we have started to generate an event by generating values of x and Q according to probabilities given by the usual factorization formula for the inclusive cross section. We wish to generate a full event.

1. Given the momenta of the photon and the hadron, q^μ and P^μ, we generate a value N_u for the number of final-state partons and then a final-state partonic configuration $(l_1,\ldots,l_{N_u})$ with the density extracted from eq. (4.42)

$$
\prod_{j=1}^{N_u} \left[\frac{d^6 l_j}{(2\pi)^6} \, J_I(l_j^2) \right] \frac{\hat{H}_{N_u}\,\Phi(k,P)}{\sigma[1]} \Delta_{N_u}\,. \tag{6.1}
$$

   This can either be done directly, or by separate steps of generating a massless configuration $(\hat{l}_1,\ldots,\hat{l}_{N_u})$ together with values of k^-, $\vec{k}_T$ and M_1, $\ldots M_{N_u}$, as at the end of section 2, with probability densities extracted from from eq. (4.37). In the second case we need to calculate the true partonic variables.

   The hard scattering is evaluated perturbatively with the renormalization and factorization scales μ and μ_J chosen to be of order Q, to avoid large logarithms.

2. For each final-state parton j that is above some threshold of perturbative invariant mass $l_j^2 > Q_0^2$, generate a final-state parton shower using the algorithm given in [2] in its own CM frame. Then boost it to the true l_j. Since we already know the exact momentum of the parton, we should omit the step of finding the invariant mass.

   Otherwise the parton is left unchanged for nonperturbative hadronization.

3. If the initial-state parton is above the threshold of perturbative invariant mass $|k^2| > Q_0^2$, generate its parton shower. This is done by looping back to the first step, with the photon momentum q replaced by the parton momentum $-k$, and with the factorization formulas changed to the ones for the pCf.

   Otherwise the parton is left unchanged for nonperturbative hadronization.

4. Once all remaining partons that are above the perturbative threshold have been showered, apply nonperturbative hadronization.

This recursive algorithm for event generation reduces the whole problem to a repeated application of the same procedure, so that the program size is independent of the number of particles generated and the computational resources are linear in the number of particles.

7. Zero- and one-loop corrections to coefficient functions

In this section we give the coefficient functions for the structure function and the parton correlation function up to one-loop order.

7.1 Structure function

For the hard scattering coefficients for the DIS structure function we could work directly with diagrams for the hard-scattering component of factorization, i.e., diagrams for the subgraph labeled "H" in a decomposition like figure 4. However, we will find it convenient to put the graphs in the context of their use in the factorization theorem. Thus we consider graphs of the form of figure 6, where we have attached a full amplitude for the pCf Φ to the graphs from which the one-loop hard-scattering coefficient function is calculated.

There are certain complications in the interpretation of these graphs. Observe first that there is some apparent double counting. For example, the ladder graph (b) also implicitly appears in (a), with the rung l_2 of (b) being inside the Φ subgraph of (a). We actually mean that we consider each term in figure 4 or 6 not as a graph as such but as a decomposition of a graph for the purposes of examining its behavior: a graph can have multiple decompositions. We will indeed sum over the different decompositions, and the subtraction procedure in the hard scattering will take care of the double counting.

Next, whereas we have used an explicit graphical factor Φ to represent the connection to the target proton, we have omitted corresponding fragmentation factors for the connection of the final-state partons to the final-state hadrons. A more thorough presentation

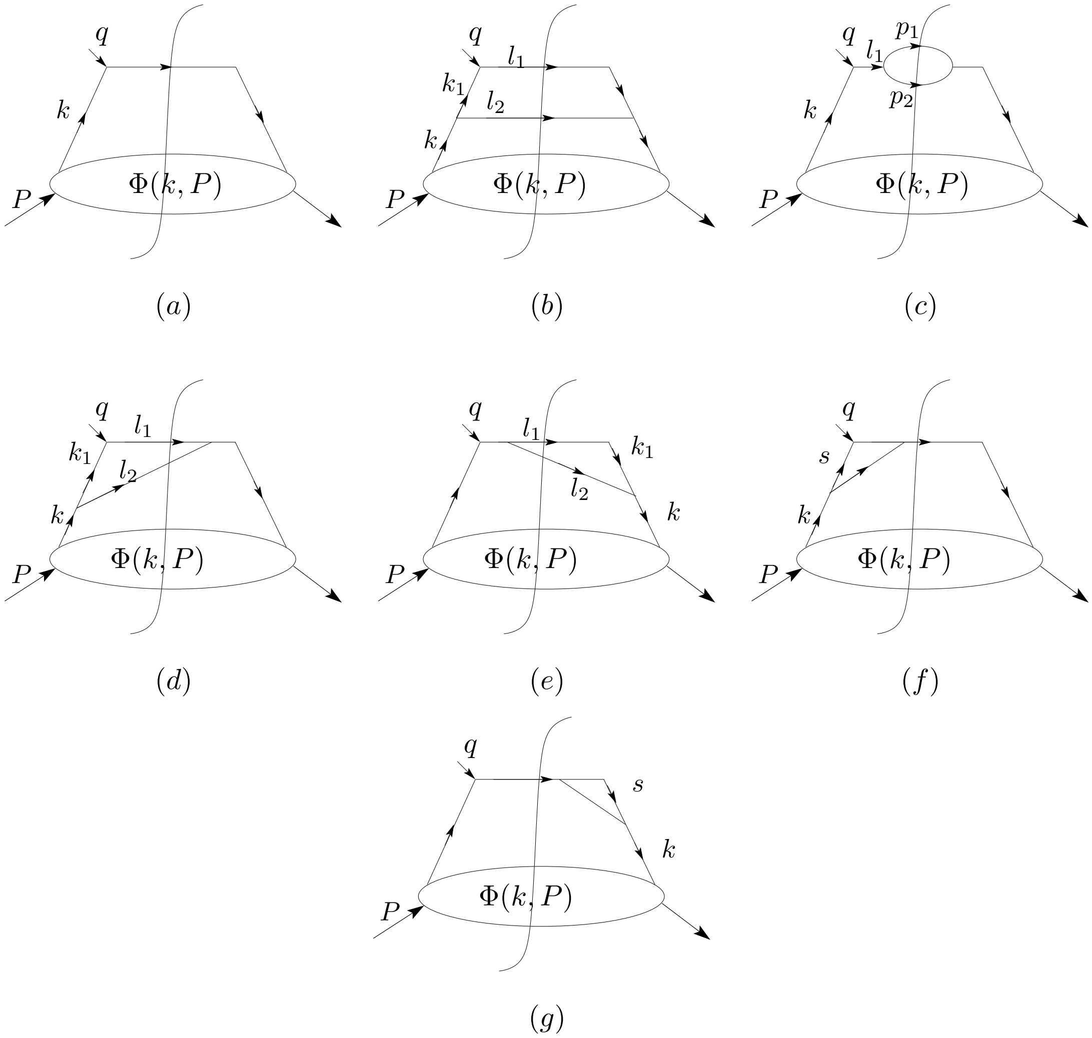

Figure 6: One-loop graphs for DIS.

would have done this, but our focus in this paper is on issues associated with initial-state partons.

Also, observe that self-energy corrections on the incoming parton line are always included in the pCf. Therefore we omit from our considerations decompositions such as figure 7; the self-energy on the incoming line does *not* contribute to our calculation of the one-loop hard-scattering coefficients.

Similarly self-energy corrections on final-state parton lines, as in figure 8, are always to be considered as part of the hadronization, and these self-energies make no direct contribution are therefore disregarded in the calculation of the hard-scattering coefficients. But observe that in the application to an algorithm for a MC event generator, there are normalization factors associated with final-state parton in the definition of the properly normalization hard scattering *cross section* $d\hat{\sigma}_{N_u}$. These are the factors of Z in eq. (4.38), and their calculation, given in [2], does indeed include a consideration of self-energy graphs.

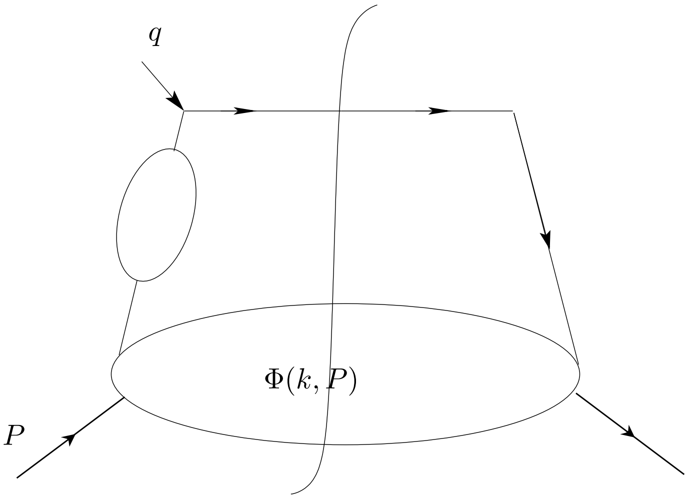

Figure 7: A graph like this with an initial-state parton self energy is not used.

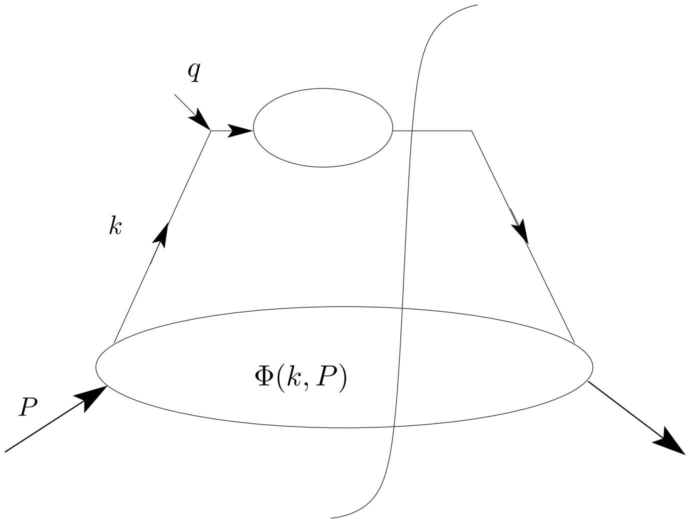

Figure 8: In a graph like this the final-state self energy contributes to the hadronization of the final-state parton, and such graphs are disregarded in computing the hard-scattering coefficient.

7.1.1 Zero-loop graph (a)

Graph (a) has a tree graph coupled to the full pCf. The hard scattering contribution corresponds to the region where the incoming parton line k is collinear to the target. Comparing the graph with the general form of factorization, (4.31) and (4.37), we see that the hard scattering coefficient is

$$\hat{H}_1 = e^2 \left[1 + O(g^2)\right], \tag{7.1}$$

with a corresponding cross section

$$d\hat{\sigma}_1 = \frac{\pi}{q^-} \delta(\hat{k}^+ - q^+) K e^2 \left[1 + O(g^2)\right] = \frac{\pi x P^+}{Q^2} \delta(\hat{k}^+ - xP^+) K e^2 \left[1 + O(g^2)\right]. \tag{7.2}$$

7.1.2 Graph (b)

Now we treat the ladder graph, (b) of figure 6. We always suppose k is collinear to the target. Then there are two leading regions. In region R_1, k_1, l_2 and k are almost collinear to the target, with $m^2 \sim |k^2| \leq |k_1^2| \ll Q^2$. In region R_2, k_1 and l_2 are at wide angle, with[12] $m^2 \sim |k^2| \ll |k_1^2| \leq Q^2$.

First we must remember that, as noted earlier, graph (b) also appears as a particular case of graph (a). We constructed an approximation to graph (a), and obtained the lowest-order hard scattering. In terms of graph (b), this approximation corresponds to the case that momentum k_1 is almost collinear to the target, i.e., it is a good approximation in region R_1. Therefore our analysis of the regions for (b) will be designed to obtain an appropriately subtracted hard scattering corresponding to the other region.

This is done in terms of the approximation operations for the two regions of graph (b), which correspond to different projections of the parton momenta on massless momenta. The projection for region R_1 we denote by[13] $(\hat{k}_1,\ \hat{l}_1,\ l_2,\ k)$ with jacobian Δ_1, and the projection for region R_2 we denote by $(\tilde{k}_1,\ \tilde{l}_1,\ \tilde{l}_2,\ \tilde{k})$ with jacobian Δ_2. The values are:

$$\begin{aligned}
\hat{k}_1 &= \left(-q^+, 0, \vec{0}_T\right), \\
\hat{l}_1 &= \left(0, q^-, \vec{0}_T\right),
\end{aligned} \tag{7.3}$$

$$\Delta_1 = 1 + \frac{k_1^-}{q^-} = 1 + 2xP^+ \frac{k_1^-}{Q^2}, \tag{7.4}$$

and

$$\begin{aligned}
\tilde{l}_1 &= \left(\frac{(\vec{l}_{1T} - \vec{l}_{2T})^2}{4(q^- + l_1^- - l_2^-)}, \frac{q^- + l_1^- - l_2^-}{2}, \frac{\vec{l}_{1T} - \vec{l}_{2T}}{2} \right), \\
\tilde{l}_2 &= \left(\frac{(\vec{l}_{1T} - \vec{l}_{2T})^2}{4(q^- + l_2^- - l_1^-)}, \frac{q^- + l_2^- - l_1^-}{2}, \frac{\vec{l}_{2T} - \vec{l}_{1T}}{2} \right), \\
\tilde{k}_1 &= \left(-q^+ + \frac{(\vec{l}_{1T} - \vec{l}_{2T})^2}{4(q^- + l_1^- - l_2^-)}, \frac{-q^- + l_1^- - l_2^-}{2}, \frac{\vec{l}_{1T} - \vec{l}_{2T}}{2} \right), \\
\tilde{k} &= \left(-q^+ + \frac{(\vec{l}_{1T} - \vec{l}_{2T})^2}{2[(q^-)^2 - (l_1^- - l_2^-)^2]}, 0, \vec{0}_T \right),
\end{aligned} \tag{7.5}$$

$$\Delta_2 = \frac{l_1^- l_2^-}{(l_1^- - k^-/2)(l_2^- - k^-/2)} = \frac{4 l_1^- l_2^-}{[(q^-)^2 - (l_1^- - l_2^-)^2]}. \tag{7.6}$$

Before approximation, graph (b) is represented by:

$$\int \Gamma_b = \int \frac{d^6 k}{(2\pi)^6} \Phi(k,P) \int \frac{d^6 k_1}{(2\pi)^6} \frac{e^2 g^2}{(k_1^2 - m^2)^2} \int \frac{d^5 \vec{l}_1}{(2\pi)^5 2E_1} (2\pi)^6 \delta^{(6)}(l_1 - k_1 - q) \times$$

[12] Here, for convenience of power counting, we choose m to be the typical small scale. In general all we need is some scale s which is much smaller than the hard scattering scale Q. That means, the virtuality of the external partons of the hard subgraph is of order s^2. If s is much larger than the hadronization scale Λ then we can recursively apply factorization to the jets generated by those external partons.

[13] Notice that no approximation is made on k or l_2 in this region; these momenta are considered as part of the pCf.

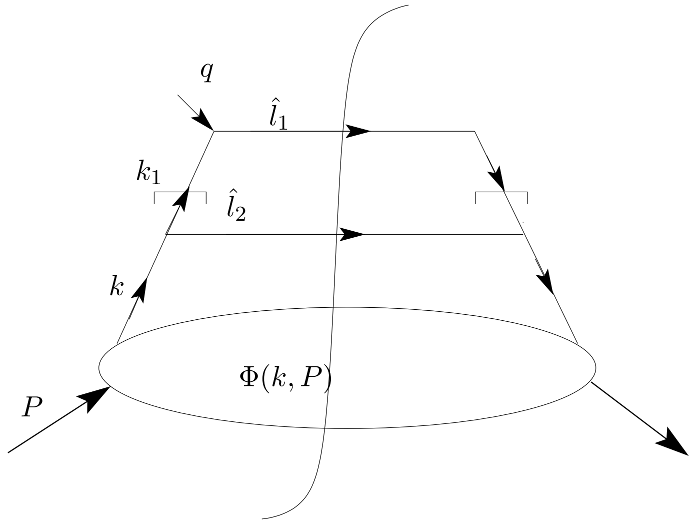

Figure 9: Approximation for region R_1 of figure 6a. The small square brackets indicate where the collinear approximation of eq. (4.14) is made.

$$\times \int \frac{d^5 \vec{l}_2}{(2\pi)^5 2E_2} (2\pi)^6 \delta^{(6)}(l_2 - k + k_1)\,. \tag{7.7}$$

Our goal is to apply the general decomposition by regions that we gave in eqs. (4.4), (4.25) with our chosen definition of the approximation operator T_R. We will construct terms $\Gamma_{b1} = C_{R1}(\Gamma_b)$ and $\Gamma_{b2} = C_{R2}(\Gamma_b)$ and will explicitly check that the sum is a good approximation to the original graph:

$$\Gamma_b = \Gamma_{b1} + \Gamma_{b2} + O\left(\frac{|\Gamma_b| m^2}{Q^2}\right) \tag{7.8}$$

everywhere that k is collinear to P.

Region R_1**.** In this region, $|k_1^2|$ is $O(m^2)$, so l_2 and k are almost parallel. The corresponding approximation, symbolized in figure 9, is

$$\int \Gamma_{b1} = \int T_{R1} \Gamma_b = \int \frac{d\hat{k}^+}{2\pi} \int \frac{d^5 k}{(2\pi)^5} \Phi^{(1)}(k_1, P) dL(\hat{l}_1; q + \hat{k}_1) \hat{H}_1(\hat{l}_1, \hat{k}_1; q, m = 0)\,, \tag{7.9}$$

with

$$\Phi^{(1)}(k_1, P) = \int \frac{d^6 k}{(2\pi)^6} \Phi(k, P) dL(l_2; k - k_1) \frac{e^2 g^2}{(k_1^2 - m^2)^2} \tag{7.10}$$

representing a one-rung iteration of the pCf. The approximated momenta are defined by eq. (7.3). The result in eq. (7.9) gives of course just an example of a lowest order hard scattering, and we already know that the contribution to the hard scattering factor is $\hat{H}_1(\hat{l}_1, \hat{k}_1; q, m = 0) = e^2$.

When constructing the subtractions for region R_2, we need the formula for Γ_{b1} in terms of the original phase space integration:

$$\int \Gamma_{b1} = \int \frac{d^6 k}{(2\pi)^6} \Phi(k, P) \int \frac{d^6 k_1}{(2\pi)^6} dL(l_1; k_1 + q) dL(l_2; k - k_1) \Delta_1 \frac{e^2 g^2}{(k_1^2 - m^2)^2}\,. \tag{7.11}$$

Also we need to verify that Γ_{b1} is indeed a good approximation to Γ_b in region R_1:

$$\int (\Gamma_b - \Gamma_{b1}) = \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^6k_1}{(2\pi)^6}\, dL(l_1; k_1+q)\, dL(l_2; k-k_1) \times \\ \times \frac{e^2g^2}{(k_1^2-m^2)^2}(1-\Delta_1)\,. \tag{7.12}$$

Since $1-\Delta_1 = k_1^-/q^-$, which is small in R_1, the desired suppression follows.

Region R_2. In this region, both of the final-state partons l_1 and l_2 are at wide angle. The approximation for the whole graph in this region is

$$\int T_{R2}\Gamma_b = \int \frac{d\tilde{k}^+ dk^- d^4\vec{k}_T}{(2\pi)^6}\Phi(k,P)\, dL(\tilde{l}_1,\tilde{l}_2; q+\tilde{k}; m=0) \times \\ \times \int \frac{d^6\tilde{k}_1}{(2\pi)^6}\delta^{(6)}(\tilde{l}_2 - (\tilde{k}_1 - \tilde{k}))H_2\,. \tag{7.13}$$

Here $H_2 = e^2g^2/(\tilde{k}_1^2)^2$, while $\tilde{k}$, $\tilde{k}_1$, $\tilde{l}_1$, and $\tilde{l}_2$ are given by (7.5).

The hard scattering factor H_2 has a collinear singularity when $\tilde{l}_2$ and $\tilde{k}$ become parallel. However, this is in the smaller region R_1, where Γ_{b1} is a good approximation, and for which we are required to apply a subtraction, according to the general formula (4.25). Therefore we define Γ_{b2} to be the approximation of the original graph Γ_b *minus* the approximation in R_1, i.e., Γ_{b1}. Then the singularity in the collinear region will be canceled point-by-point in the integration space:

$$\begin{aligned}\int \Gamma_{b2} &= \int V\left(-\frac{\tilde{k}_1^2}{m^2}\right) T_{R2}(\Gamma_b - \Gamma_{b1}) \\ &= \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^6k_1}{(2\pi)^6} \int \frac{d^5\vec{l}_1}{(2\pi)^5 2E_1}(2\pi)^6\delta^{(6)}(l_1 - q - k_1) \\ &\times \int \frac{d^5\vec{l}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}(l_2 - k + k_1)\, V\left(-\frac{\tilde{k}_1^2}{m^2}\right)\frac{e^2g^2}{(\tilde{k}_1^2)^2}(1-\Delta'_{1b})\Delta_2\,.\end{aligned} \tag{7.14}$$

Here Δ'_{1b} is the jacobian in the subtraction term with the momenta k, etc. replaced by the approximated momenta $\tilde{k}_1$, etc., that are appropriate for region R_2, i.e.,

$$\begin{aligned}\Delta'_{1b} &= \frac{\tilde{l}_1^-}{\tilde{l}_1^- - \tilde{k}_1^-} = \frac{l_1^- - k^-/2}{l_1^- - k^-/2 + \tilde{l}_2^-} = \frac{l_1^- - k^-/2}{l_1^- - k_1^-} \\ &= \Delta_1 - \frac{k^-}{2(l_1^- - k_1^-)} = \Delta_1\left[1 + O\left(\frac{m^2}{Q^2}\right)\right].\end{aligned} \tag{7.15}$$

The jacobian Δ_2 is given by eq. (4.22) with $N_u = 2$. The collinear veto factor $V(\tilde{k}_1/m)$, defined in eq. (4.29), is used to remove a spurious *nonleading* collinear divergence in the differential cross section due to the factor $H_2 = e^2g^2/(\tilde{k}_1^2)^2$. Since Γ_{b2} is designed to handle momentum configurations far from the collinear region, the precise functional form of the veto factor in the collinear region is irrelevant to physics. Any other function which is zero when $|\tilde{k}_1^2| \ll m^2$ and unity when $|\tilde{k}_1^2| \gg m^2$ would be equally good.

It follows from eq. (7.14) that the contribution of the one-rung ladder to the hard scattering is

$$V\bigg(-\frac{\tilde{k}_1^2}{m^2}\bigg)\frac{e^2g^2}{\big(\tilde{k}_1^2\big)^2}(1-\Delta'_{1b})\,. \tag{7.16}$$

Validity of approximations. We can check that Γ_{b2} is indeed power suppressed in region R_1.

$$\Gamma_{b2}=VT_{R2}(\Gamma_b-\Gamma_{b1})=O\left(|\Gamma_b-\Gamma_{b1}|\frac{(k_1^2)^2}{(\tilde{k}_1^2)^2}\right)=O\left(|\Gamma_b|\frac{|k_1^2|}{Q^2}\right)\overset{R_1}{=}O\left(|\Gamma_b|\frac{m^2}{Q^2}\right), \tag{7.17}$$

where we have used eq. (7.12) to estimate the size of $\Gamma_b-\Gamma_{b1}$. In region R_2, where $|k^2|$ is the typical scale that we ignored and $|k_1^2|$ is the typical hard scale, we can see the following is true.

$$\Gamma_{b2}-(\Gamma_b-\Gamma_{b1})=O\left(\left|\frac{k^2}{k_1^2}\right||\Gamma_b-\Gamma_{b1}|\right)=O\left(\frac{m^2}{Q^2}|\Gamma_b|\right). \tag{7.18}$$

Combining the error estimates in different regions we can see that Γ_{b1} and Γ_{b2} indeed satisfy our requirement that $\Gamma_b=\Gamma_{b1}+\Gamma_{b2}+O(|\Gamma_b|m^2/Q^2)$ in all regions.

7.1.3 Graph (c)

Next we treat graph (c) of figure 6. The treatment is essentially the same as that of graph (b), except that the subtraction is about final-state fragmentation. The graph has two leading regions. For its smallest region, for which we use the notation R_{c1}, p_1 and p_2 are almost parallel. In region R_{c2}, p_1 and p_2 are at wide angle. Since final state subtractions have been treated in detail in [2], we will give the result directly without details. The major difference is in the projections to massless momenta in DIS and e^+e^-. Again, we use symbols like $\hat{l}_1$ for approximated momenta in region R_{c1} and symbols like $\tilde{l}_1$ for those in region R_{c2}.

The graph can be represented as

$$\int\Gamma_c=\int\frac{d^6k}{(2\pi)^6}\Phi(k,P)\int\frac{d^5\vec{p}_1}{(2\pi)^5 2E_1}\int\frac{d^5\vec{p}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}((q+k)-(p_1+p_2))\times$$
$$\times\frac{e^2g^2}{(q+k)^2-m^2}\,. \tag{7.19}$$

Region R_{c1}. In region R_{c1} of graph (c), Γ_c is approximated as

$$\begin{aligned}\int\Gamma_{c1}&=\int T_{R_{c1}}\Gamma_c\\&=\int\frac{d\hat{k}^+}{2\pi}\int\frac{dk^-d^4\vec{k}_T}{(2\pi)^5}\Phi(k,P)\int\frac{d^5\hat{\vec{l}}_1}{(2\pi)^5 2|\hat{\vec{l}}_1|}(2\pi)^6\delta^{(6)}(\hat{l}_1-q-\hat{k})\times\\&\quad\times\int\frac{dM_1^2}{2\pi}\int\frac{d^5\vec{p}_1}{(2\pi)^5 2E_1}\int\frac{d^5\vec{p}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}(l_1-p_1-p_2)\frac{e^2g^2\Theta(M_1^2/\mu_J^2)}{[(p_1+p_2)^2-m^2]^2}\,,\\&=\int\frac{d^6k}{(2\pi)^6}\Phi(k,P)\int\frac{d^5\vec{p}_1}{(2\pi)^5 2E_1}\int\frac{d^5\vec{p}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}(q+k-p_1-p_2)\times\end{aligned}$$

$$\times \frac{e^2 g^2 \Theta(M_1^2/\mu_J^2)\Delta_{c1}}{(q+k)^2 - m^2}\,, \tag{7.20}$$

where the jacobian is $\Delta_{c1} = l_1^-/(l_1^- - k^-)$ and the mass of parton l_1 is $M_1^2 = l_1^2 = (p_1+p_2)^2$.

This value is in fact a full pCf times the lowest-order hard scattering times an order g^2 approximation to final-state showering.[14] Thus it is to be found as a contribution to full factorized cross section with the lowest-order hard scattering; there is no new contribution to the hard scattering.

Region R_{c2}. For region R_{c2} of graph (c), we have the approximation

$$\begin{aligned}\int T_{R_{c2}}\Gamma_c = \int \frac{d\tilde{k}^+}{2\pi}\int \frac{dk^- d^4\vec{k}_T}{(2\pi)^5}\Phi(k,P)\int \frac{d^5\tilde{\vec{p}}_1}{(2\pi)^5|\tilde{\vec{p}}_1|}\int \frac{d^5\tilde{\vec{p}}_2}{(2\pi)^5|\tilde{\vec{p}}_2|}(2\pi)^6 \times \\ \times \delta^{(6)}(\tilde{p}_1+\tilde{p}_2-q-\tilde{k})\frac{e^2g^2}{[(\tilde{p}_1+\tilde{p}_2)^2]^2}\,,\end{aligned} \tag{7.21}$$

which is divergent in the collinear region R_{c1}. After subtraction we have

$$\begin{aligned}\int \Gamma_{c2} &= \int V\left(\frac{\tilde{M}_1^2}{m^2}\right)T_{R_{c2}}(\Gamma_c - \Gamma_{c1}) \\ &= \int \frac{d^6k}{(2\pi)^6}\Phi(k,P)\int \frac{d^5\vec{p}_1}{(2\pi)^5 2E_1}\int \frac{d^5\vec{p}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}(q+k-p_1-p_2)\times \\ &\quad \times V\left(\frac{\tilde{M}_1^2}{m^2}\right)\frac{e^2g^2}{[(\tilde{p}_1+\tilde{p}_2)^2]^2}\left[1-\Theta\left(\frac{(\tilde{p}_1+\tilde{p}_2)^2}{\mu_J^2}\right)\Delta'_{1c}\right]\Delta_{c2}\,,\end{aligned} \tag{7.22}$$

where Δ'_{1c} is the jacobian calculated with massless momenta for region R_2

$$\Delta'_{1c} = \frac{\tilde{p}_1^- + \tilde{p}_2^-}{\tilde{p}_1^- + \tilde{p}_2^- - k^-} = \frac{p_1^- + p_2^- - k^-}{p_1^- + p_2^- - 2k^-} = \Delta_1\left(1+\frac{O(k^-)}{l_1^-}\right) = \Delta_1\left(1+O\left(\frac{m^2}{Q^2}\right)\right). \tag{7.23}$$

Δ_{c2} is defined using eq. (4.22) with $N_u = 2$. It is easy to check that $\Gamma_c = \Gamma_{c1} + \Gamma_{c2} + O(m^2/Q^2)$ in all regions.

It follows that there is the following contribution to the hard scattering $\hat{H}_2$:

$$V\left(\frac{\tilde{M}_1^2}{m^2}\right)\frac{e^2g^2}{[(\tilde{p}_1+\tilde{p}_2)^2]^2}\left[1-\Theta\left(\frac{(\tilde{p}_1+\tilde{p}_2)^2}{\mu_J^2}\right)\Delta'_{1c}\right]. \tag{7.24}$$

7.1.4 Graphs (d) and (e)

Graphs (d) and (e) of figure 6 are hermitian conjugates of each other. They each have only one leading region, where the two final-state particles are at wide angle. We have

$$\begin{aligned}\int \Gamma_d = \int \frac{d^6k}{(2\pi)^6}\Phi(k,P)\int \frac{d^6k_1}{(2\pi)^6}\int \frac{d^5\vec{l}_1}{(2\pi)^5 2E_1}\int \frac{d^5\vec{l}_2}{(2\pi)^5 2E_2}(2\pi)^6\delta^{(6)}(l_1-q-k_1)\times \\ \times(2\pi)^6\delta^{(6)}(l_2-k+k_1)\frac{e^2g^2}{(k_1^2-m^2)\,[(q+k)^2-m^2]}\end{aligned} \tag{7.25}$$

[14]Note that, whereas we drew graph (a) with an explicit factor for the initial-state interactions, we did not explicitly give it its corresponding final-state factor.

$$\int \Gamma_e = \text{H.c.} \int \Gamma_d = \int \Gamma_d\,, \tag{7.26}$$

$$\int \Gamma_{d1} = \int T_R \Gamma_d = \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^5\vec{l}_1}{(2\pi)^5 2E_1} \int \frac{d^5\vec{l}_2}{(2\pi)^5 2E_2} \times$$
$$\times (2\pi)^6 \delta^{(6)}(l_1 + l_2 - q - k) \frac{e^2 g^2 \Delta_2}{(q+\hat{k})^2(\hat{l_2}-\hat{k})^2}\,, \tag{7.27}$$

where Δ_2 and the approximated momenta $(\hat{k},\ \hat{l_2},\ \ldots)$ are defined using eq. (4.22) and eq. (4.19) with $N_u = 2$.

The contribution to the hard scattering of these two graphs is

$$2\frac{e^2 g^2}{(q+\hat{k})^2(\hat{l_2}-\hat{k})^2}\,. \tag{7.28}$$

7.1.5 Virtual corrections: graphs (f) and (g)

Graph (f) and (g) in figure 6 provide vertex corrections to one-parton production. They have only one leading region, where the loop momentum is hard. There is also a UV divergence in the loop integral which we renormalize using the $\overline{\text{MS}}$-scheme, so that

$$\int \Gamma_f = \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^5\vec{l}_1}{(2\pi)^5 2E_1}\,(2\pi)^6\delta^{(6)}(l_1 - q - k) \times \tag{7.29}$$
$$\times \int \frac{d^6k_1}{(2\pi)^6}\frac{e^2g^2}{(k_1^2 - m^2)[(q+k_1)^2 - m^2][(k-k_1)^2 - m^2]} + \text{UV counterterm}\,,$$

$$\int \Gamma_g = \text{H.c.} \int \Gamma_g \tag{7.30}$$

$$\int \Gamma_{f1} = \int T_R \Gamma_f = \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^5\vec{l}_1}{(2\pi)^5 2E_1}\,(2\pi)^6\delta^{(6)}(l_1 - q - k)\Delta_1$$
$$\times \int \frac{d^6k_1}{(2\pi)^6}\frac{e^2g^2}{k_1^2(q+k_1)^2(\hat{k}-k_1)^2} + \overline{\text{MS}}\ \text{counterterm}\,,$$
$$= \int \frac{d^6k}{(2\pi)^6}\Phi(k,P) \int \frac{d^5\vec{l}_1}{(2\pi)^5 E_1}\,(2\pi)^6\delta^{(6)}(l_1 - (q+k)) \times$$
$$\times \Delta_1 \frac{e^2g^2}{128\pi^3}\left(\ln\frac{Q^2}{\mu^2} - 3\right). \tag{7.31}$$

Note that the vertex correction depends on the renormalization scale μ; this just shows that the "current" $j = 1/2[\phi^2]$ has an anomalous dimension. The renormalization group of the current can be found in [2]. The anomalous dimension is canceled by the anomalous dimension of e^2.

7.1.6 Total

Combining the $O(g^2)$ hard-scattering coefficients from graphs (b) to (e) in figure 6, we then have the hard-scattering coefficient for $\gamma^* + \text{parton} \to p_1 + p_2$:

$$\hat{H}_2 = V\left(-\frac{(\tilde{k}-\tilde{p}_2)^2}{m^2}\right)\frac{e^2g^2}{[(\tilde{k}-\tilde{p}_2)^2]^2}(1-\Delta'_{1b}) + \frac{2e^2g^2}{(q+\tilde{k})^2(\tilde{p}_2-\tilde{k})^2} +$$

$$+V\left(\frac{(\tilde{p}_1+\tilde{p}_2)^2}{m^2}\right)\frac{e^2g^2}{[(\tilde{p}_1+\tilde{p}_2)^2]^2}\left[1-\Theta\left(\frac{(\tilde{p}_1+\tilde{p}_2)^2}{\mu_J^2}\right)\Delta'_{1c}\right]+O(e^2g^4)\,, \quad (7.32)$$

with Δ'_{1b} and Δ'_{1c} defined in eq. (7.15) and (7.23), respectively, and V being the veto function that cuts out the singularities. The approximated momenta $\tilde{p}_1$, *etc.* are defined in section 4.3 with $N_u = 2$.

From graphs (a), (f) and (g) we find the one-parton term with one-loop corrections:

$$\hat{H}_1 = e^2\left[1+\frac{g^2}{64\pi^3}\left(\ln\frac{Q^2}{\mu^2}-3\right)+O(g^4)\right]. \quad (7.33)$$

7.2 Parton correlation function

In our model theory, the current for the "photon" is just a scalar vertex with two parton lines. Graphs for the parton correlation function are the same as for the structure function multiplied by:

- Two *full* parton propagators at momentum k, i.e., a factor $1/(k^2)^2$ at leading order.
- A factor of g^2 instead of the factor e^2 associated with the vertex for $e^2[\phi^2]/2$.
- A factor $-2\pi/k^2$ to remove the prefactor $Q^2/(2\pi)$ in our definition of the structure function — see eq. (2.4).

8. Conclusions and future work

We have shown that in a full treatment of a Monte-Carlo event generator the derivation of standard factorization is not sufficient. The physics in standard factorization is indeed unchanged. What is at issue is the validity of the identification, for example, of the momenta of the external partons of a lowest-order hard scattering with the naive parton model values, eq. (2.5). It is not just that showering of the parton k in figure 3 gives it nonzero values k^- and $\vec{k}_T$, for that is implicit already in a correct definition of a parton density. The problem that forces us to change views is that k^+ is changed by the showering of the other parton. We have shown how these issues can be handled by the use of a new factorization theorem that uses unintegrated parton correlation functions, and in a model theory we have derived an appropriate theorem.

In the new factorization theorem, we consider the structure function differential in the full final state, or more conveniently we consider the structure function with an arbitrary weight function applied to the final state. The form of the factorization enables a systematic first level of decomposition of the partonic structure to be made, differential in exact parton kinematics. The hard-scattering has a systematic perturbative expansion to all orders of perturbation theory. Instead of ordinary parton densities, the theorem uses parton correlation functions.

Factorization also applies to the parton correlation functions, thus enabling a systematic recursive decomposition of the fully exclusive parton structure of generated events. The formalism covers arbitrarily nonleading logarithms with the use of an expansion of

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the coefficient functions in powers of the coupling, without logarithms, and with the use of suitable renormalization-group transformations. It lends itself very naturally to a recursive implementation suitable for a MCEG.

The new factorization theorems that we have just summarized give a decomposition of the final-state into its exclusive components. We also need separate factorization theorems that give the cross section and the parton correlation function integrated over final states; these enable the first pair of factorization theorems to provide conditional probabilities to be used for the MCEG. For the cross section integrated over all final states, factorization is just ordinary factorization, which expresses it in terms of ordinary parton densities. We then showed that an almost identical argument gives a very similar result for the parton correlation function.

The formalism is completed with the aid of renormalization group and DGLAP evolution, that allow perturbative calculations to be done at the appropriate scales in each component of the different factorizations. The same ideas apply to final-state showering, but in a simpler way [2].

Using the subtractive procedure proposed in [2] to define higher order corrections to the kernels, we constructed an algorithm for a MCEG for DIS (including initial-state parton showering), in which coefficient functions for both the hard scattering and the showering can be obtained at arbitrarily non-leading order. The subtractions are applied point-by-point in momentum space, so that the hard scattering coefficients and the evolution kernels are normal functions, unlike the singular generalized functions that appear in inclusive calculations from conventional factorization.

Because of the subtractions, the hard scattering coefficients are not necessarily positive. An EG with weights is in general to be expected [1]. An algorithm for weighted EG was proposed in [12].

Although our derivations only apply to non-gauge theories, and there is much work to be done to extend our work to full QCD, we believe our work indicates how one should approach the full QCD problem. One considers the regions of momentum space that dominate but without relying on any unitarity cancellation of a soft region. This is like the situation in factorization with transverse-momentum-dependent (TMD) parton densities [11, 13], which work applies to inclusive cross sections that are sensitive to parton transverse momentum. In QCD (and any gauge theory), two complications arise. The first is that there is an extra non-canceling soft factor to be treated. The second is that to be gauge-invariant, the parton correlation functions must be defined with Wilson lines in the operators, and factorization requires [13]–[17] the Wilson lines to be in certain directions. Examination of the general techniques used in these works suggests that they should continue to apply to the weighted cross sections we use to understand the final state. Naturally the appropriate factorization must involve gauge-invariant unintegrated parton correlation functions, so that parton momenta can be treated exactly. There are extra techniques already available for handling the extra complications, for example the Collins-Soper evolution equation for the direction dependence of the Wilson lines. So many of the elements needed to extend our results on MCEGs to full QCD are available, and simply need to be adjusted and applied in the new context.

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Recently Collins and Metz [17] have performed a detailed general analysis of the directions of the Wilson lines needed for factorization; technically it depends on an understanding of the allowed directions of certain contour deformations in virtual graphs that are needed in the proof of factorization. In the case of both e^+e^--annihilation and of DIS, their analysis of TMD factorization does not have any requirement that the cross section be inclusive. Since such cross sections involve a soft factor, no unitary cancellations are needed to get factorization. But one will probably need multi-parton soft factors to accompany multi-parton production processes, with corresponding new nonperturbative features.

It is the use of TMD factorization that saves the need for the unitary cancellation of the soft factor that happens in non-TMD factorization.

But as Collins and Metz explain, TMD factorization in hadron-hadron processes, like Drell-Yan, continues very critically to need a unitarity cancellation. Therefore, the limits to MCEGs are likely to be more severe here. This suggests that a full treatment will need a discussion of spectator-spectator interactions that goes far beyond a normal factorization framework. Elements of this are undoubtedly already present in some existing EGs like PYTHIA.

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